연세경제연구 제 IX 권 제 1 호
Yonsei Economic Studies, Vol. IX, No. 1 (March 2002)

Applications of the Pareto Distribution

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The so-called Pareto distribution is named after the celebrated Italian social scientist Vilfredo Pareto (1848 ~ 1923). He applied it to his analysis of income distribution. The mathematical form of the Pareto distribution is simple[1])

$$f(x) = \frac{\theta x_0^\theta}{x^{\theta+1}}, \quad 0 < \theta, \quad 0 < x_0 < x \, ; \; \mu_x = \frac{\theta x_0}{\theta - 1} \quad \text{for} \quad \theta > 1 \, ;$$

$$\sigma_x^2 = \frac{\theta x_0^2}{(\theta-1)^2(\theta-2)} \quad \text{for} \quad \theta > 2 \, ; \quad E(x^\gamma) = \frac{\theta x_0^\gamma}{\theta - \gamma} \quad \text{for} \quad \gamma < \theta \, ;$$

Moment generation function does not exist.

1. i . An Old Duchess deserted by her husband brings the engagement ring, a legendary diamond, to her most trusted jeweler. He humbly answers: "Your Highness, I have to find a buyer first. Anyway, I will get you at least 10

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1) Mood, Graybill and Boes [1] pp. 542 ~ 543.

million dollars. However, I am very much reluctant to tell you how much I can get you beyond that." The Duchess looks puzzled. The jeweler hastily explains: "You know I am usually precise. In this case, I cannot tell you how much you may expect. You may expect an astronomical price on this. I am on an unprecedented mission of selling a priceless gem."

Neither the schrewd loyal jeweler nor the poor rich Duchess really grasps the situation. The Duchess needs at least 15 million dollars to soothe her wounded pride and pay off her debts. Heartily wishing her good luck, we calculate the probability for her.

Let W in 10 million dollars be the price the diamond fetches. Then W is a random variable. The specifications are: $W > 1$, μ_W is infinite, $f(w)$ is monotonically decreasing, and the mode is realized at $W = 1$. Among distribution which satisfy the above specifications, $f(w) = w^{-2}$, $w > 1$, is the simplest. The required probability is

$$\int_{1.5}^{\infty} w^{-2} dw = -w^{-1}]_{1.5}^{\infty} = 0.6667$$

A half Cauchy distribution $f(w) = \frac{2}{\pi} \frac{1}{1+(w-1)^2}$, $w > 1$, also satisfies the specification. If this distribution is adopted, the required probability is

$$\int_{0.5}^{\infty} \frac{2}{\pi} \frac{1}{1+(w-1)^2} dw = 0.7047$$

The difference between the two results is acceptable.

ii. The jeweler responds otherwise. "Your Highness, I can get you at least 10 million dollars, which is also the most likely amount the jewel can fetch.

But the amount can be much higher, Indeed, you may expect 20 million dollars *on average.*"

The Duchess seems perplexed.

"What do you mean by *on average*?"

The jeweler is dismayed.

"I myself don't know exactly what I mean. But you once said to me, 'I had good servants and bad ones. However, they were not so bad *on averge.*' Both of us have some idea about *on average*, haven't we?"

The Duchess faintly blushes.

"I see we use words without knowing their exact meanings."

Now the jewel is legendary but not priceless in the jeweler's eyes. How much will the Duchesse's prospect deteriorate? It is statistician's professional duty to clarify a stochastic situation and give an answer, if required.

Still a Pareto distribution well describes the situation.

However, θ is specified differently.

$$\mu_W = 2 = \frac{\theta \cdot 1}{\theta - 1}, \quad \theta = 2; \quad \text{hence,} \quad f(x) = \frac{2 \cdot 1}{w^2}, \quad w > 1$$

The required probability is:

$$\int_{1.5}^{\infty} \frac{2.1}{w^3}\, dw = 0.4444.$$

We feel sorry for the Duchess.

$$f(w) = \lambda e^{-\lambda(w-1)}, \quad w > 1$$

also reasonably describes the situation.

$$\mu_W = 2 = \int_1^\infty w\lambda e^{-\lambda(w+1)}dw = 1+\frac{1}{\lambda}, \quad \lambda = 1.$$

The required probability is:

$$\int_{1.5}^\infty e^{-(w-1)}dw = 0.60653.$$

We prefer the exponential to the Pareto on behalf of the lonely, proud Duchess.

2. We present here a stochastic model of 'limit' pricing. If a monopolist sets his price too high and reaps an excessive profit, other firms may enter and share the market. Hence, the monopolist should set his price at a moderate level in order to prevent new entries. The so called 'limit' price is the maximum level at which a monopolist can set his price persistently without inducing any entry.

Firm A has recently developed a new product called K, and may enjoy a monopolistic position, though a substitute for K can be developed. Firm B wants to buy it on a long term basis at a reasonable fixed price. However, B has no idea what price A will charge. In fact, B has no information about either A's pricing or potential competition. B can only guess: A's average cost plus 'normal' profit is anywhere between \$20 and \$25; A prevents new entries at any cost, so as to prevent new entries. A sets his price most likely at the average cost plus a 'normal' profit. But A may charge an exorbitant price if he is sure that nobody dare enters the market regardless whatever A may do. From B's viewpoint, what is the probability that the price A will charge will be less than \$30?

Under the stated conditions: Let Y be A's average cost plus the 'normal' profit, then $f(y) = \frac{1}{5}$, $20 < y < 25$; let X be the price A will charge, then

$$f(x \mid y) = \frac{y}{x^2}, \quad y < x < \infty.$$

$$f(x) = \int_{20}^{x} \frac{y}{x^2} \cdot \frac{1}{5}\, dy = \frac{1}{10} - \frac{40}{x^2}, \qquad \text{for } 20 < x < 25$$

$$= \int_{20}^{25} \frac{y}{x^2} \cdot \frac{1}{5}\, dy = \frac{22.5}{x^2}, \qquad \text{for } 20 < x < \infty$$

$$\int_{20}^{\infty} f(x)\,dx = \int_{20}^{25}\left(\frac{1}{10} - \frac{40}{x^2}\right)dx + \int_{25}^{\infty} \frac{22.5}{x^2}\,dx$$

$$= 0.1 + 0.9 = 1$$

The required probability is:

$$\int_{20}^{25}\left(\frac{1}{10} - \frac{40}{x^2}\right)dx + \int_{25}^{30} \frac{22.5}{x^2}\,dx = 0.25$$

3. Firm A desperately needs a certain kind of machine. However, A cannot make it by itself. A intends to procure it from outside by bidding. A's cost accounting team estimates the manufacturing cost of the machine plus a normal profit is 10 million dollars. Instead A solicits bids with the proviso that the bidding will be revoked if the minimum bidding price is unacceptably high. A has good reason to assume that the number of responding bidders, Y, is a positive Poisson variable with mean equal to 5. Hence,

$$f(y) = \frac{\frac{e^{-5}5^y}{y!}}{1-e^{-5}}, \quad y = 1, 2, 3, \cdots$$

A conjectures about the bidders' strategies. From his viewpoint: If there is only one bidder, he will fully take advantage of his monopolistic position, and hence his bidding price can be indefinitely large, so A cannot even expect what the bidding price will be; if there are two bidders, A can assume there will be collusion and betrayal between the two, so the expected minimum bidding price is exorbitantly higher than 10 million dollars, but A still has no idea how far the actual minimum bidding price will deviate from the expected minimum bidding price; as the number of the responding bidders increases beyond two, the minimum bidding price is expected to be nearer and nearer to 10 million dollars, and the actual minimum bidding price will deviate less and less from the expected minimum bidding price; the bidders face the constraint that they must understand that A may revoke the bids if the minmum bidding price is too high. Let X (in 10 million dollars) be the minimum bidding price. Then

$$f(x \mid y) = \frac{y \cdot 1}{y^{y+1}}, \quad x > 1$$ [2)]

What is the probability that X is less than 1.1 from A's viewpoint?

The marginal probability density function of X is obtained as

2) For $y = 1$, μ_X is infinite; for $y = 2$, $\mu_X = 2 \cdot 1$, σ_X^2 undefined; for $y \geq 3$, $\mu_X = \frac{y}{y-1} \cdot 1$, $\sigma_X^2 = \frac{y \cdot 1^2}{(y-1)^2(y-2)}$.

The mode is set at $X =$ 10 million dollars; this is justified by the constraint.

$$f(x) = \sum_{y=1}^{\infty} \frac{y}{x^{y+1}} \frac{\frac{e^{-5}5^y}{y!}}{1-e^{-5}}$$

$$= \frac{e^{-5}}{1-e^{-5}} \frac{5}{x^2} e^{\frac{5}{x}}, \quad x > 1$$

It is easily seen;

$$\int_1^{\infty} \frac{e^{-5}}{1-e^{-5}} \frac{5}{x^2} e^{\frac{5}{x}} dx = 1$$

$$\int_1^{1.1} \frac{e^{-5}}{1-e^{-5}} \frac{5}{x^2} e^{\frac{5}{x}} dx = \frac{e^{-5}}{1-e^{-5}} \left[e^5 - e^{\frac{5}{1.1}}\right]$$

It may be asked: if the minimum bidding price is a Pareto random variable, what distribution describes the behavior of the bidders from A's view point? Let $\{x_1, x_2, x_3, \cdots, x_n\}$, be a random sample of size n from a population with the simplest Pareto distribution, $f(s) = s^{-2}$, $s \geq 1$.

If $t = \min\{x_1, x_2, x_3, \cdots, x_n\}$, then $f(t) = nt^{-(n+1)}$, $n > 1$,

$$\mu_T = \frac{n}{n-1} \quad \text{for} \quad n \geq 2, \quad \sigma_T^2 = \frac{n^2}{(n-1)(n-2)}, \quad n \geq 3$$ [3)]

The above derivation simply means: the bidders play in the most simple 'Pareto ground', and they play independently of each other on account of the constraint; as the number of the bidder increases, the outcome of the bidding becomes more competitive.

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3) In general: Let $f(u) = \frac{ku_0^k}{u^{k+1}}$, $k > 0$, $u > u_0 > 0$, and let $\{x_1, x_2, x_3, \cdots, x_n\}$ be a random sample from the population.

If $v = \min\{x_1, x_2, x_3, \cdots, x_n\}$ then $f(v) = \frac{knu_0^{kn}}{v^{kn+1}}$, $kn > 0$, $v > v_0$.

▣ *References* ▣

1. Mood, A. M., Graybill, F. A. and D. C. Boes, *Introduction to the Theory of Statistics*, 3rd ed., Tokyo: McGraw-Hill, 1974, pp. 542～543.