연세경제연구 제 Ⅷ 권 제 1 호
Yonsei Economic Studies, Vol. Ⅷ, No. 1 (March 2001)

<강의자료 Teachers' Corner>

# Applications of the $\chi^2$ Distribution

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1. The longevity of a certain lighting device is a memory-less random variable, and its mean is inversely proportional to how far an essential component deviates from the origin of a two-dimensional Cartesian space inside the device. The placement of the component is a random phenomenon. Let $(X, Y)$ be the position of the component. It is assumed that $X$ and $Y$ are statistically independent and

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty, \qquad \text{and}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \qquad -\infty < y < \infty$$

Hence, if $U = X^2 + Y^2$, $f(u) = \frac{1}{2} e^{-\frac{u}{2}}$, $u > 0$.

On the other hand, the longevity of the device, $T$, is formulated as follows:

$$f(t|u) = ue^{-ut}$$

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$$\begin{aligned} f(t) &= \int_0^\infty f(t|u) f(u) du \\ &= \int_0^\infty ue^{-ut} \frac{1}{2} e^{-\frac{u}{2}} du \\ &= \frac{1}{2}\left(\frac{1}{2}+t\right)^{-2}, \qquad t>0 \end{aligned}$$

What is the probability that the device lasts more than 10 time-units?

$$\int_{10}^\infty \frac{1}{2}\left(\frac{1}{2}+t\right)^{-2} dt = \frac{1}{21}$$

What, if the component is stochastically placed at the origin of a three-dimensional Cartesian space inside the device?

Let $V = X^2 + Y^2 + Z^2$.

Then $f(v) = \dfrac{1}{\left(\frac{1}{2}\right)! 2^{\frac{3}{2}}} v^{\frac{1}{2}} e^{-\frac{v}{2}}, \qquad v > 0$

$$\begin{aligned} f(t|v) &= ve^{-vt}, \qquad t > 0, \qquad \text{and} \\ f(t) &= \int_0^\infty f(t|v) f(v) dv \\ &= \int_0^\infty ve^{-vt} \frac{1}{\left(\frac{1}{2}\right)! 2^{\frac{3}{2}}} v^{\frac{1}{2}} e^{-\frac{v}{2}} dv \\ &= \frac{3}{2} \cdot \frac{1}{2^{\frac{3}{2}}} \left(\frac{2}{2t+1}\right)^{\frac{5}{2}}, \qquad t > 0 \end{aligned}$$

It is easy to see $\displaystyle\int_0^\infty \frac{3}{2} \cdot \frac{1}{2^{\frac{3}{2}}} \left(\frac{2}{2t+1}\right)^{\frac{5}{2}} dt = 1$

2. We deal here with the simplest stochastic model of a learning process. Wilhelm Tell aims at the center of a target 60 feet away. The target is a

circle of $\sqrt{2}$ feet radius.

Tell may miss the center horizontally or vertically. Gessler, der Vogt, demands that Tell hit the target in 5 shots. However, Gessler is generous in his own way. He lets Tell have sufficient rest and nourishment between shots so that he won't lose his initial strength at all, as he keeps on shooting. And learning effects are substantial, so the randomness of his shooting decreases, as he shoots repeatedly. For instance, his sense of distance becomes more accurate or he gets better accustomed to surrounding conditions such as the wind, as his shooting experience accumulates.

Let $X_n$ and $Y_n$ be the horizontal and the vertical error in feet, respectively on the $n$th shot. Assume

$$X_n \sim Normal\left(0, 1\,(feet)^2 \times \left(\frac{9}{10}\right)^{n-1}\right)$$

$$Y_n \sim Normal\left(0, 1\,(feet)^2 \times \left(\frac{9}{10}\right)^{n-1}\right)$$

and $X_n$ and $Y_n$ are statistically independent.

If Tell meets Gessler's demand, Tell will go scot free. Otherwise, Tell will be jailed. What is the probability that he goes scot free?

The probability of success on the $n$th shot:

$$\Pr(X_n^2 + Y_n^2 < \sqrt{2}^2) = \Pr\left(\frac{X_n^2 + Y_n^2}{\left(\frac{9}{10}\right)^{n-1}} < \frac{\sqrt{2}^2}{\left(\frac{9}{10}\right)^{n-1}}\right)$$

$$= \int_0^{\frac{2}{(\frac{9}{10})^{n-1}}} \frac{1}{2} e^{-\frac{v}{2}} dv$$

$$= 1 - e^{-(\frac{10}{9})^{n-1}}$$

The required probability:

$$1-e^{-1}+e^{-1}(1-e^{-\frac{10}{9}})+e^{-1}e^{-\frac{10}{9}}(1-e^{-(\frac{10}{9})^2})$$
$$+e^{-1}e^{-\frac{10}{9}}e^{-(\frac{10}{9})^2}(1-e^{-(\frac{10}{9})^3})$$
$$+e^{-1}e^{-\frac{10}{9}}e^{-(\frac{10}{9})^2}e^{-(\frac{10}{9})^3}(1-e^{-(\frac{10}{9})^4})$$
$$=1-e^{-1}e^{-\frac{10}{9}}e^{-(\frac{10}{9})^2}e^{-(\frac{10}{9})^3}e^{-(\frac{10}{9})^4}$$

3. An essential component is placed at the origin of a two-dimensional Cartesian space inside a machine. However, the placement may deviate from the origin in any direction by any amount. Let $X$ and $Y$ be the deviations from the origin on the first and second axes, respectively. Then $X$ and $Y$ are random variables. Assume $X$ and $Y$ are independent, and $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $f(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$.

On the other hand : if the component is not precisely placed at the origin, the machine may produce defective products. Suppose the number of defectives is a Poisson variable with mean equal to the square of the deviation. What is the probability that there will be more than 10 defectives?

Let $U=X^2+Y^2$. Then, $f(u)=\frac{1}{2}e^{-\frac{u}{2}}$, $u>0$. Let $T$ be the number of the defectives. Then, $f(t|u)=\frac{e^{-u}u^t}{t!}$, $t=0,1,2,\cdots$.

$$f(t)=\int_0^\infty f(t|u)f(u)du$$
$$=\int_0^\infty \frac{e^{-u}u^t}{t!}\cdot\frac{1}{2}e^{-\frac{u}{2}}du=\frac{1}{2}\cdot\left(\frac{2}{3}\right)^{t+1}$$
$$t=0,1,2,\cdots$$

The required probability is $\sum_{t=10}^{\infty}\frac{1}{2}\left(\frac{2}{3}\right)^{t+1}$.

**4.** A special task force plans to destroy an enemy's fortress with a missile. The landing of the missile is a stochastic phenomenon due to human factors such as imperfect eyesight or surrounding conditions such as the wind. Place the center of the fortress at the origin of two dimensional Cartesian space, and let $(X, Y)$ denote the landing spot.

Assume $X \sim Normal(0, u)$, $Y \sim Normal(0, u)$, and $X$ and $Y$ are statistically independent. $U$ is a random variable, which is at least 1 and can be indefinitely large. Suppose $f(u) = u^{-2},\ u > 1$.[1)]

If $X^2 + Y^2 < 1$, the fortress will be destroyed, what is the probability that the force will accomplish its mission?

Let $X^2 + Y^2 = V$.

Then $f(v) = \frac{1}{2} e^{-\frac{v}{2}},\ v > 0$

$$\Pr(X^2 + Y^2 < 1) = \Pr\left(\frac{X^2 + Y^2}{U} < \frac{1}{U}\right)$$

$$= \int_1^\infty \int_0^{\frac{1}{u}} \frac{1}{2} e^{-\frac{v}{2}} u^{-2}\, dv du$$

$$= 2e^{-\frac{1}{2}} - 1 = 0.23$$

**5.** We invert here a non-parametric test into an equivalent parametric test.

Mr. $K$., a man in charge of training sharp shooters, suspect that $P$, one of

1) This is a Pareto distribution with infinite mean and undefinable variance. Suppose a probability density function, $f(u)$ has the following properties : $U$ has the minimum $u_0 = 1$, and can be indefinitely large, that is, $u_0 = 1 < u < \infty$ ; $u_o = 1$ is the most likely, namely $u_o = 1$ is the mode ; as u increases, it is less likely, symbolically $f'(u) < 0$ ; no information is derivable on mean and variance. Then $f(u) = u^{-2},\ u > 1$ reasonably describes the situation.

his trainees, slacks in shooting. $P$ should be tested. $P$ takes aim at the origin of a circle with radius equal to 1m, positioned 400m away from the target.

The simplest stochastic model to do the job is as follows:

$$f(w_i) = \theta^{w_i}(1-\theta)^{1-w_i}$$

$0<\theta<1$, the hit rate

$w_i = 1$ if hits the target

$= 0$ if misses the target.

Take a sample of size $n=20$. $U=\sum_{i=1}^{20} w_i$ 2) will be a sufficient test statistics for $\theta$. Suppose the null hypothesis $H_0$: $\theta=0.8$, and the alternative hypothesis $H_1$: $\theta=0.5$. Set the level of significance, $\alpha$ at 0.05. Then, the best critical region consists of $0\leq u\leq 12$; Type I error probability = 0.03214; The power of the test = 0.86841.

An equivalent parametric test may be formulated as follows: The deviation of a shot from the center of the target can be thought of as an observation error. Let $X$, $Y$ be the horizontal and vertical deviations from the center, respectively. Assume $X$ and $Y$ are independent. Then,

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}},$$

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} \quad \text{and}$$

$\sigma$ is determined as follows:

For $\theta=0.8$, $0.8 = P(X^2 + Y^2 < 1)$

2) Learning or wearing down effects are assumed away.

$$= P\left(\frac{X^2+Y^2}{\sigma^2} < \frac{1}{\sigma^2}\right)$$

$$= \int_0^{\frac{1}{\sigma^2}} \frac{1}{2} e^{-\frac{v}{2}} dv = 1 - e^{-\frac{1}{2\sigma^2}}$$

$$\sigma^2 = \frac{1}{\log 25} = 0.31066\,;$$

for $\theta = 0.5$, $\sigma^2 = \frac{1}{\log 4} = 0.72134$

Take a sample of size $n = 15$. $V = \sum_{i=1}^{15} \frac{X_i^2 + Y_i^2}{\sigma^2}$ will be a sufficient test statistics for $\sigma^2$. $H_0 : \sigma^2 = \frac{1}{\log 25}$; $H_1 : \sigma^2 = \frac{1}{\log 4}$; $\alpha = 0.05$. The best critical region is on the right side. The best critical region will be $V > 43.773$ and the power of the test will be 0.93 even for a sample size $15 < 20$. The superiority of a parametric test over an equivalent non-parametric test in terms of the power of the test is confirmed. This is nothing surprising. A parametric test uses relevant information more than an equivalent non-parametric test.