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Analyzing an Axially Symmetric Noncausal 2-D AR Process Using a Causal 2-D AR Model

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If the autocorrelation function of a stationary noncausal 2-D AR process is symmetric about both axes, the 2-D AR process is said to be axially symmetric. If an axially symmetric noncausal 2-D AR process is bi-causal, there exists a causal 2-D AR process having the same autocorrelations as the given noncausal 2-D AR process. Hence an axially symmetric noncausal 2-D AR process can be analyzed through the corresponding causal 2-D AR model.

I. Introduction

Let $\{y_{s,t}\}$ be a stationary random field, and define its autocovariance function (ACVF) and autocorrelation function (ACRF) by $\sigma(j, k) = \mathrm{Cov}(y_{s+j, t+k}, y_{s,t})$ and $\rho_{j,k} = \sigma(j, k)/\sigma(0,0)$, respectively. It is clear that an ACRF is symmetric about the origin, i.e., $\rho_{j,k} = \rho_{-j,-k}$. If the ACRF is symmetric about both axes, i.e.,

$$\rho_{j,k} = \rho_{-j,k} = \rho_{j,-k} = \rho_{-j,-k} \qquad (j, k = 0, 1, \cdots) \tag{1}$$

it is said to be axially-symmetric (e.g. Martin [4]). If the ACRF of a noncausal 2-D AR process is axially symmetric, there exists a causal 2-D AR process having the same autocorrelations as the given noncausal 2-D AR process. This autocorrelation equivalence relation (AER) is utilized for analysis of a noncausal 2-D AR process

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through a causal 2-D AR model.

Ⅱ. Autocorrelation Equivalence Relation

Consider a stationary 2-D random field $\{y_{s,t}\}$ satisfying a noncausal 2-D AR model

$$\sum_{j=-p_1}^{p_2} \sum_{k=-q_1}^{q_2} \beta_{j,k} y_{s-j,t-k} = -v_{s,t}, \qquad \beta_{0,0} = -1 \tag{2}$$

where $p_1, p_2, q_1, q_2 \geq 0$, $p_1 p_2 q_1 q_2 \neq 0$, and $\{v_{s,t}\}$ is a sequence of 2-D i.i.d. random variables with means 0 and variances $\sigma^2 > 0$. Besag [1] names the model (2) a noncausal simultaneous AR model. It should be assumed, due to symmetry of the ACRF, that $p_1 = p_2$, $q_1 = q_2$ and $\beta_{j,k} = \beta_{-j,-k}$ $(j = 0, \pm 1, \cdots, \pm p_1, k = 0, \pm 1, \cdots, \pm q_1)$. Otherwise, the noncausal 2-D AR model is not identifiable. A noncausal 2-D AR process satisfying these conditions is said to be symmetric. When the ACRF of a noncausal 2-D AR process is axially symmetric, we assume $p_1 = p_2 = q_1 = q_2 = p$ and $\beta_{j,k} = \beta_{-j,k} = \beta_{j,-k} = \beta_{-j,-k}$ $(j, k = 0, \cdots, p)$ to avoid an identifiability problem. Such a noncausal 2-D AR process is said to be axially symmetric. In this paper we confine ourselves to an axially symmetric noncausal AR process satisfying

$$\beta(B_1, B_2)\, y_{s,t} = v_{s,t} \tag{3}$$

where B_1 and B_2 are backshift operators satisfying $B_1^d B_2^R y_{s,t} = y_{s-j,t-k}$, and

$$\begin{aligned}\beta(z_1, z_2) = 1 &- \sum_{j=1}^{p} \beta_{j,0}(z_1^j + z_1^{-j}) - \sum_{k=1}^{p} \beta_{0,k}(z_2^k + z_1^{-k}) \\ &- \sum_{j=1}^{p} \sum_{k=1}^{p} \beta_{j,k}(z_1^j z_2^k + z_1^{-j} z_2^k + z_1^j z_2^{-k} + z_2^{-j} z_2^{-k})\end{aligned} \tag{4}$$

Let $\gamma(z_1, z_2)$ be a real 2-D polynomial satisfying

$$\gamma(z_1, z_2) = -\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \gamma_{j,k} z_1^{-j} z_2^{-k}, \qquad \gamma_{0,0} = -1 \tag{5}$$

$$\gamma(z_1, z_2) \neq 0 \quad \text{for} \quad |z_1| \geq 1 \quad \text{or} \quad |z_2| \geq 1 \tag{6}$$

If there exists a real 2-D polynomial $\gamma(z_1, z_2)$ satisfying (5), (6), and

$$\beta(z_1, z_2) = c_1\gamma(z_1, z_2)\,\gamma(z_1^{-1}, z_2^{-1}) \tag{7}$$

where c_1 is a positive constant, and if $\gamma^2(z_1, z_2)$ has a finite number of terms, then the axially symmetric noncausal 2-D AR process (3) is said to be bi-causal. Its spectrum is

$$S_y(\lambda_1, \lambda_2) = \frac{1}{c_1^2}\frac{\sigma^2}{(2\pi)^2}\frac{1}{|\gamma^2(e^{i\lambda_1}, e^{i\lambda_2})|^2} \tag{8}$$

Clearly, $S_y(\lambda_1, \lambda_2)$ in (8) is also the spectrum of a causal 2-D AR process satisfying

$$\gamma^2(B_1, B_2)y_{s,t} = u_{s,t} \tag{9}$$

where $\{u_{s,t}\}$ is a sequence of 2-D i.i.d. random variables. Thus, the axially symmetric noncausal 2-D AR process (3) and the causal 2-D AR process (9) have the same autocorrelations. This is called the AER of the 2-D AR processes. Let

$$\alpha(z_1, z_2) = -\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha_{j,k}z_1^{-j}z_2^{-k} = \gamma^2(z_1, z_2), \qquad \alpha_{0,0} = -1 \tag{10}$$

The coefficients $\{\gamma_{j,k}\}$ can be calculated using the formula

$$\gamma_{j,k} = \frac{1}{2}(\alpha_{j,k} + \sum_{(a,b)=G_{j,k}}\gamma_{a,b}\,\gamma_{j-a,k-b}) \tag{11}$$

where $G_{j,k} = \{(a, b)\,|\,a = 0, 1, \cdots, j, b = 0, 1, \cdots, k\} - \{(0, 0), (j, k)\}$.

It is more convenient and more efficient to handle a causal 2-D AR process than a noncausal one. Thus, a bi-causal axially symmetric noncausal 2-D AR process would rather be analyzed through the corresponding causal one based on the AER.

Ⅲ. Applications

Since there exists no fundamental theorem of algebra for 2-D polynomials, it is difficult to derive a closed form of the ACRF of a 2-D AR process. The ACRF's of some causal 2-D AR models are studied in literature such as Besag [1], Tory and Pickard [5]. The ACRF of a bi-causal axially symmetric noncausal 2-D AR process can be easily derived based on the AER, when that of the corresponding causal 2-D AR process is given. As an example, consider an axially symmetric noncausal 2-D AR process satisfying

$$\begin{aligned} y_{s,t} = & \frac{3}{10}(y_{s+1,t} + y_{s-1,t}) + \frac{2}{5}(y_{s,t+1} + y_{s,t-1}) \\ & -\frac{3}{25}(y_{s+1,t+1} + y_{s+1,t-1} + y_{s-1,t+1} + y_{s-1,t-1}) + v_{s,t} \end{aligned} \tag{12}$$

where $\{v_{s,t}\}$ is a sequence of 2-D i.i.d. random variables. It can be easily verified that

$$\beta(z_1, z_2) = \frac{18}{25}\gamma(z_1, z_2)\,\gamma(z_1^{-1}, z_2^{-1}) \tag{13}$$

where

$$\gamma(z_1, z_2) = \left(1-\frac{1}{3}z_1^{-1}\right)\left(1-\frac{1}{2}z_2^{-1}\right) \tag{14}$$

Equations (13) and (14) show that the noncausal 2-D AR process (12) is bi-causal. The noncausal 2-D AR process (12) has the same ACRF as a causal 2-D AR process satisfying

$$\left(1-\frac{1}{3}B_1\right)^2\left(1-\frac{1}{2}B_2\right)^2 y_{s,t} = u_{s,t} \tag{15}$$

where $\{u_{s,t}\}$ is a sequence of 2-D i.i.d. random variables. The Yule-Walker equations of the causal 2-D AR process (15) is

$$\rho_{j,k} = \frac{2}{3}\rho_{j-1,k} - \frac{1}{9}\rho_{j-2,k} + \rho_{j,k-1} - \frac{2}{3}\rho_{j-1,k-1} + \frac{1}{9}\rho_{j-2,k-1} - \frac{1}{4}\rho_{j,k-2} + \frac{1}{6}\rho_{j-1,k-2} - \frac{1}{36}\rho_{j-2,k-2} \quad (j>0 \text{ or } k>0) \tag{16}$$

It can be verified that the solution of the partial difference equation (16) is

$$\rho_{j,k} = a^{|j|}\left(1 + \frac{1-a^2}{1+a^2}|j|\right) b^{|k|}\left(1 + \frac{1-b^2}{1+b^2}|k|\right) \quad (j, k = 0, \pm 1, \pm 2, \cdots) \tag{17}$$

where $a=1/3$ and $b=1/2$. The AER shows that $\{\rho_{j,k}\}$ in (17) is the ACRF of the noncausal 2-D AR process (12).

When the ACRF of an axially symmetric noncausal 2-D AR process is given, we can calculate the coefficients $\{\beta_{j,k}\}$ based on the AER. As an example, assume that the ACRF (17) is given. Denote by $\alpha_{l,l}(z_1, z_2)$ the 2-D z-transform of a causal 2-D AR model of orders l and l having the ACRF. Condition (6) implies that the causal 2-D AR model (9) is stable. Hence we apply the order-recursive algorithm in Choi [3] to (17), and obtain

$$\alpha_{l,l}(z_1, z_2) = \left(1-\frac{2}{3}z_1^{-1}+\frac{1}{9}z_1^{-2}\right)\left(1-z_2^{-1}+\frac{1}{4}z_2^{-2}\right), \quad (l=2, 3, \cdots) \tag{18}$$

Thus, the ACRF (17) is from a causal 2-D AR model satisfying

$$\left(1-\frac{2}{3}B_1+\frac{1}{9}B_1^2\right)\left(1-B_2+\frac{1}{4}B_2^2\right) y_{s,t} = u_{s,t} \tag{19}$$

where $\{u_{s,t}\}$ is a sequence of 2-D i.i.d. random variables. It can be verified using (11) that

$$\gamma(z_1, z_2) = \left(1-\frac{1}{3}z_1^{-1}\right)\left(1-\frac{1}{2}z_2^{-1}\right) \tag{20}$$

The AER shows that the ACRF (17) is also from a noncausal 2-D AR model

$$\left(1-\frac{1}{3}B_1\right)\left(1-\frac{1}{3}B_1^{-1}\right)\left(1-\frac{1}{2}B_2\right)\left(1-\frac{1}{2}B_2^{-1}\right)y_{s,t}=v_{s,t} \qquad (21)$$

where $\{v_{s,t}\}$ is a sequence of 2-D i.i.d. random variables.

The AER can be utilized in noncausal 2-D AR modeling. Let $\{y_{s,t} | s = 1, 2, \cdots, S, t = 1, 2, \cdots, T\}$ be a ST-realization of an axially symmetric noncausal 2-D AR process. To fit an axially symmetric noncausal 2-D AR model to the realization, we first find a suitable causal 2-D AR model for it. Then, we determine the former based on the AER. Therefore, it is not necessary to develope new modeling theories for an axially symmetric noncausal 2-D AR process such as maximum likelihood estimation, order determination methods and diagnostic checking methods, which are difficult to accomplish.

Ⅳ. Conclusion

If an axially symmetric noncausal 2-D AR process is bi-causal, then there exists a causal 2-D AR process having the same autocorrelation function as the noncausal 2-D AR process. This AER makes it possible to circumvent some difficulties of the axially symmetric noncausal 2-D AR model analysis.

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