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<강의자료 Teachers' Corner>

Applications of the Power Distribution

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The power distribution simply means

$$
\begin{aligned}
f(x) &= \alpha x^{\alpha-1}, \quad \alpha > 0, \quad 0 < x < 1 \\
&= 0, \qquad \textit{otherwise}
\end{aligned}
$$

This is perhaps a misnomer. However, it is retained until a better name is found.

$$
\mu_x = \frac{\alpha}{\alpha+1}, \quad \mu_x \rightarrow 1, \quad as \quad \alpha \rightarrow \infty, \quad \text{and}
$$

$$
\mu_x \rightarrow 0, \quad as \quad \alpha \rightarrow 0
$$

$$
\sigma_x^2 = \frac{\alpha}{(\alpha+1)^2(\alpha+2)} \quad \sigma_x^2 \rightarrow 0, \quad as \quad \alpha \rightarrow \infty, \quad \text{and}
$$

also $\sigma_x^2 \rightarrow 0, \ as \ \alpha \rightarrow 0$.

The power distribution is a Beta distribution, and we are familiar with it. Recall : Let $\{x_1, x_2, \cdots, x_n\}$ be a random sample from a continuous uniform population on a unit interval. Then, the probability density function of $U = \max\{X_1, \cdots, X_n\}$ is given by

$$
\begin{aligned}
f(u) &= nu^{n-1}, \quad 0 < u < 1 \\
&= 0, \qquad \textit{otherwise}
\end{aligned}
$$

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1. Firm A has lost an engineer. It must fill the position immediately. It can do this by attracting applicants through a newspaper advertisement and by interviewing them. A knows, from past experience, that the number of applicants is a Poisson variable with the mean equal to 5, and there is at least one applicant. However, there is no guarantee that any of the applicants will be qualified. And yet, the more applicants there are, the better A' s chances are that there will be at least one qualified applicant.

Let θ be the probability of getting at least one qualified applicant, and x be the number of applicants. Then, x follows what is called the positive Poisson distribution whose probability mass function is

$$f(x) = \frac{\frac{e^{-5} \cdot 5^x}{x!}}{1-e^{-5}}, \quad x = 1, 2, 3, \cdots$$
$$= 0, \qquad otherwise$$

And $$f(\theta | x) = x\,\theta^{x-1}, \quad 0 < \theta < 1$$
$$= 0, \qquad otherwise$$

well describes the situation.

On the other hand : whether or not A gets at least one qualified applicant is described by

$$f(u | \theta) = \theta^u (1 - \theta)^{1-u}, \quad u = 1 \text{ if } A \text{ succeeds},$$
$$u = 0 \text{ if } A \text{ fails}$$

What is the probability that A gets at least one qualified applicant?

$$f(\theta) = \sum_{x=1}^{\infty} f(\theta | x) f(x)$$
$$= \sum_{x=1}^{\infty} x\theta^{x-1} \cdot \frac{\frac{e^{-5} \cdot 5^x}{x!}}{1-e^{-5}}$$
$$= \frac{e^{-5} \cdot 5e^{5\theta}}{1-e^{-5}}$$

$$\int_0^1 \frac{e^{-5} \cdot 5e^{5\theta}}{1-e^{-5}} d\theta = \frac{e^{-5}}{1-e^{-5}}(e^5 - 1)$$
$$= 1$$

$$\int_0^1 \theta^1(1-\theta)^{1-1}\frac{e^{-5} \cdot 5e^{5\theta}}{1-e^{-5}} d\theta = 0.8068$$

2. A paper mill manager dumps waste water into a large river. If he is caught dumping, he will be fined dearly. However, he may or may not be caught, if he dumps.

The probability that he is caught dumping is positively correlated with the intensity of his dumping per time unit.

Let θ be the probability that the manager is caught dumping, and α[1] be the intensity of his dumping per time unit. Then,

$$f(\theta) = \alpha\theta^{\alpha-1}, \quad \alpha > 0, \quad 0 < \theta < 1$$
$$= 0, \qquad \textit{otherwise}$$

reasonably describes the situation. Because $\mu_\theta \longrightarrow 1$ as $\alpha \longrightarrow \infty$, and $\sigma_\theta^2 \longrightarrow 0$, as $\alpha \longrightarrow \infty$, the manager will be surely caught, if he dumps an infinitely large amount of waste water per time unit.

How much can the manager dump if he wants the probability that his chances of being caught dumping are less than 0.25 to be more than 0.75?[2]

$$\int_0^{0.25} \alpha\theta^{\alpha-1} d\theta > 0.75$$

$$\alpha < \frac{\log 0.75}{\log 0.25} = 0.2075$$

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1) α is a real number. Suppose the authorities permit dumpings upto K tons per time unit but prohibit any excess over it. Then α will be $\frac{X-K}{K}$, where X is dumpings by the mill in tons per time unit.

2) This is not a nonsensible question. Suppose there are 5 identical urns. Urn Ⅰ contains 1 white ball and 4 black balls; Urn Ⅱ 2 white balls and 3 black balls; Urn Ⅲ 3 white balls and 2 black balls; Urn Ⅳ 4 white balls and 1 black ball; Urn Ⅴ 5 white balls. A man chooses one of the urns randomly, and draws a ball. What is the probability that the man's probability of getting a white ball is more than 0.5? Obviously the answer is $\frac{3}{5}$.

3. A man engages in moulding. His success probability itself is a stochastic phenomenon; it significantly depends on how many assistants he employs; the more assistants he employs, the better it becomes. He already has one assistant, and tries to get an additional one. But he has no idea whatsoever whether or not he will be able to get an additional one.

Let θ be the success probability, and y be the number of assistants he employs. Suppose

$$f(\theta \mid y) = y\,\theta^{y-1}, \quad 0 < \theta < 1$$
$$= 0, \qquad \textit{otherwise}$$

$$f(y) = \frac{1}{2}, \quad y = 1, 2$$
$$= 0, \qquad \textit{otherwise}$$

What is the probability that the man succeeds in 5 trials?

$$f(\theta) = \sum_{y=1}^{2} f(\theta \mid y) f(y)$$
$$= \sum_{y=1}^{2} y\theta^{y-1} \cdot \frac{1}{2}$$
$$= \frac{1}{2} + \theta$$

Let x be the number of trials required to have the first success.

$$f(x \mid \theta) = (1-\theta)^{x-1} \cdot \theta, \quad x = 1, 2, 3, \cdots$$
$$= 0, \qquad \textit{otherwise}$$

$$f(x) = \int_0^1 f(\theta, x) d\theta$$
$$= \int_0^1 (1-\theta)^{x-1} \cdot \theta \left(\frac{1}{2} + \theta\right) d\theta$$
$$= \frac{1}{2} \int_0^1 (1-\theta)^{x-1} \theta d\theta + \int_0^1 (1-\theta)^{x-1} \theta^2 d\theta$$
$$= \frac{1}{2} \cdot \frac{1!(x-1)!}{(x+1)!} + \frac{2!(x-1)!}{(x+2)!}$$

$$= \frac{1}{2}\frac{1}{x(x+1)} + 2\frac{1}{x(x+1)(x+2)}$$ [3]

$$\sum_{x=1}^{5} f(x) = 0.8929$$

4. Mr. A moulds.The outcome is, of course, of a stochastic nature. Let θ be his success probability. Then, the outcome is described by :

$$f(x) = \theta^x(1-\theta)^{1-x}, \quad x = 1, \text{ if he succeeds,}$$
$$x = 0, \text{ if he fails}$$

However, θ itself is a stochastic variable between 0 and 1, and improves as A increases the number of trials.

Let y represent the number of trials. Then,

$$f(\theta) = y\,\theta^{y-1}, \quad 0 < \theta < 1$$
$$= 0, \qquad \textit{otherwise}$$

describes A's technology level on the y th trial.

The probability that A succeeds on the first trial:

$$\int_0^1 \theta^1(1-\theta)^{1-1}\cdot 1\cdot\theta^{1-1}d\theta = \frac{1}{2} = \frac{1}{2!}$$

The probability that A fails on the first trial and succeeds on the second :

$$\left(1-\frac{1}{2}\right)\int_0^1 \theta\cdot 2\theta d\theta = \frac{1}{2}\cdot\frac{2}{3} = \frac{2}{3!}$$

The probability that A fails on upto $(n-1)$ trials and succeeds on the n th trial :

$$\frac{1}{n!}\int_0^1 \theta n\,\theta^{n-1}d\theta = \frac{n}{(n+1)!}$$

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3) $\sum_{x=1}^{\infty}\frac{1}{x(x+1)(x+2)} = \sum_{x=1}^{\infty}\left\{\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+1}\right)-\frac{1}{2}\left(\frac{1}{x+1}-\frac{1}{x+2}\right)\right\}$

$= \frac{1}{2}\cdot\left(1-\frac{1}{2}\right) = \frac{1}{4}$

$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$ is confirmed by Knopp.[4]

What is the probability that A succeeds in 3 trials?

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{12+8+3}{24} = \frac{23}{24}$$

4) Knopp, K, *Theory and Application of Infinite Series*, Hafner, New York, 1947, p. 217.