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<강의자료 Teachers' Corner>

Applications of the Exponential Distribution

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1. Suppose an unskilled labor market is so chaotically competitive that a vacancy can emerge at any moment and be filled by any applicant who happens to be at the spot: for instance, positions at fast food restaurants or convenient stores. So, an applicant should roam from place to place, and the time he has already spent in seeking a position has no impact on his job prospect.

Let X be the time necessary to find a position. Then, X is "memoryless" and can be indefinitely long.

$$\begin{aligned} f(x) &= \lambda e^{-\lambda x}, \quad \lambda > 0, \quad x > 0 \\ &= 0, \qquad \textit{otherwise} \end{aligned}$$

well describes the situation.

An aristocratic young man with a trumpet decides to get a job. Let the moment of his decision be Day 0. On average 3 days are necessary to get one. He still has some money good enough to defray 10 days' expenses. His next allowance will arrive at any time between Day 8 and Day 15. If he cannot get a job or the allowance does not arrive in time, he should hock his most treasured trumpet. What is the probability that he should visit a pawn-shop?

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By Day 15, the allowance will arrive, so the young man will not seek a job thereafter; that is, the arrival of the allowance is equivalent to getting a job. Therefore, the probability that he does not get a job in time is:

$$\frac{\int_{10}^{15} \frac{1}{3} e^{-\frac{x}{3}}\, dx}{\int_{0}^{15} \frac{1}{3} e^{-\frac{x}{3}}\, dx} = \frac{e^{-\frac{10}{3}} - e^{-5}}{1 - e^{-5}}$$

On the other hand, the probability that the allowance does not arrive in time is:

$$\int_{10}^{15} \frac{1}{7}\, dx = \frac{5}{7}$$

The probability in Question is : $\dfrac{e^{-\frac{10}{3}} - e^{-5}}{1 - e^{-5}} \cdot \dfrac{5}{7} = 0.021$

2. A North Korean agent penetrates into South Korea. His mission is to get sympathizers for North Korea as many as possible. As he constantly moves from place to place, how long he can additionally be active before he gets arrested does not depend on how long he already has been active. That is, the duration of his activity is a "memoryless" random variable, and the average duration is estimated to be about 2 years from past experience. Let Y be the duration. Then,

$$f(y) = \frac{1}{2} e^{-\frac{y}{2}}, \quad y > 0$$
$$= 0, \qquad \textit{otherwise}$$

well describes the situation.

On the other hand, the number of sympathizers for North Korea he gets is a Poisson variable with the average equal to the duration of his activity. Let X be the number of sympathizers for North Korea he gets. Then,

$$f(x \mid y) = \frac{e^{-y} y^x}{x!}, \quad x = 0, 1, 2, \cdots$$

$$= 0, \qquad otherwise$$

What is the probability that the agent gets more than 4 sympathizers?

$$f(x, y) = f(x \mid y) f(y)$$

$$= \frac{e^{-y} y^x}{x!} \cdot \frac{1}{2} e^{-\frac{y}{2}}$$

$$f(x) = \int_0^\infty f(x, y)\, dy = \frac{1}{2} \cdot \left(\frac{2}{3}\right)^{x+1}, \quad x = 0, 1, 2, \cdots$$

$$= 0, \qquad otherwise$$

$$P(X \geq 5) = 1 - \sum_{x=0}^{3} \frac{1}{2} \cdot \left(\frac{2}{3}\right)^{x+1} = \frac{32}{162}$$

3. A pusher is assigned to a large territory in a metropolitan area. He pushes around surreptitiously, and hardly meets the same customer twice. As he is constantly exposed to the danger of arrest, every moment is like a fresh start. That is, how long he already has been active does not affect how long he can be active from now on. However, he assumes the average duration of his tenure will be inversely proportional to the number of anti-drug agents keenly watchful in the territory. He is certain that there is at least 1 and up to 5, but he has no idea whatsoever how many there are.

Under the above conditions, what is the probability that he can be active for more than 2 time units?

His activity can be formulated as follows: Let X be the duration of his activity, and Y is the number of the agents. As X can be considered a "memoryless" random variable, an exponential probability density function will describe the situation. Then,

$$f(x \mid y) = y e^{-yx}, \quad x > 0$$

$$= 0, \qquad otherwise$$

and $$f(y) = \frac{1}{5}, \quad y = 1, 2, 3, 4, 5$$
$$= 0, \qquad \textit{otherwise}$$

$$f(x) = \sum_{y=1}^{5} f(x, y) = \sum_{y=1}^{5} f(x \mid y) f(y)$$
$$= \frac{1}{5} \sum_{y=1}^{5} y e^{-yx}$$

$$P(X \geq 2) = \int_{2}^{\infty} f(x)\, dx$$
$$= \int_{2}^{\infty} \frac{1}{5} \sum_{y=1}^{5} y e^{-yx}\, dx$$
$$= \frac{1}{5} (e^{-2} + e^{-4} + e^{-6} + e^{-8} + e^{-10}) = 0.031$$

4. At a certain working place such as a fast food restaurant or a convenience store, there is neither paternalism nor loyalty between the employer and an employee. And as the job is simple and manual, seniority does not count in firing: so, an employee can be fired at any time. Let X be the duration of employment. Then, X is a "memoryless" random variable from the employee' s view point. And

$$f(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}, \quad \lambda > 0, \quad x > 0$$
$$= 0, \qquad \textit{otherwise}, \text{ where } \lambda \text{ is the mean employment time}$$

well describes the situation.

On the other hand, let θ be the probability that an employee immediately gets a new job after being fired, and θ is assumed to be constant; and Y be the random variable which represents the number of job searches until one fails. Thus, $Y = 1$ means that his first job search fails, and the probability of $Y = 1$ is $(1-\theta)$; $Y = 2$ means that his first job search succeeds but that his second one fails, and the probability of $Y = 2$ is $\theta(1 - \theta)$; $Y = 3$ means that his first and second job searches succeed but that his third job search fails, and the probability of the event is $\theta^2(1 - \theta)$; $\cdots$.

Let $U = X_1 + X_2 + \cdots + X_Y$. Then,

$$f(u \mid y) = \frac{1}{(y-1)!\,\lambda^y} \cdot u^{y-1} \cdot e^{-\frac{u}{\lambda}}, \quad u > 0$$

$$= 0, \qquad otherwise$$

$$f(u) = \sum_{y=1}^{\infty} f(u, y) = \sum_{y=1}^{\infty} f(u \mid y)\, f(y)$$

$$= \sum_{y=1}^{\infty} \frac{1}{(y-1)!\,\lambda^y} \cdot u^{y-1} \cdot e^{-\frac{u}{\lambda}} \cdot \theta^{y-1}(1-\theta)$$

$$= \frac{1-\theta}{\lambda} e^{-\frac{u(1-\theta)}{\lambda}}, \quad u > 0$$

$$= 0, \qquad otherwise$$

Suppose $\lambda = 5$ time units and $\theta = \frac{3}{4}$. What is the probability that an employee remains in employment for more than 20 time units.

$$\int_{20}^{\infty} \frac{1-\frac{3}{4}}{5} e^{-\frac{u\left(1-\frac{3}{4}\right)}{5}} du = e^{-1}$$

5. In an intrigue and crime ridden open city a dangerous epidemic has broken out. Those who have just arrived and those who have been residing there for quite a long time are equally vulnerable to the virus. Anyway, if one stays there long enough he will contract the disease. On average 7 days elapse before it incapacitates new arrivals.

Let X be the number of days during which a new arrival can be active without being incapacitated by the disease. Then, X is a "memory-less" random variable, and

$$f(x) = \frac{1}{7} e^{-\frac{x}{7}}, \quad x > 0$$

$$= 0, \qquad otherwise$$

describes the situation.

Agent A of Country K who is on a special mission is dispatched into the city. He initiates the operation and leaves the task for Agent B to follow upon the latter' s arrival.

Agent *B* arrives at anytime between Day 5 and Day 10. In order to take over the task, he should arrive before Agent *A* becomes incapacitated. *A* and *B* together should be active more than 15 days to accomplish the given task. What is the probability of their success?

First of all, *A* should not be incapacitated before *B* arrives. Let *Y* be *B*'s arrival time. Then, the corresponding probability is

$$\int_5^{10}\int_y^{\infty} \frac{1}{7} e^{-\frac{x}{7}} \cdot \frac{1}{5}\, dxdy = \frac{7}{5}\left(e^{-\frac{5}{7}} - e^{-\frac{10}{7}}\right) = 0.35$$

Next, let *U* be the duration during which *B* can be active. Then,

$$f(u) = \frac{1}{7} e^{-\frac{u}{7}}, \quad u > 0$$

$$= 0, \qquad otherwise$$

$$P(Y + U > 15) = \int_0^5 \int_{15-y}^{\infty} \frac{1}{5} \cdot \frac{1}{7} e^{-\frac{u}{7}}\, dudy$$

$$= \frac{7}{5} \cdot e^{-\frac{15}{7}} \cdot [e^{\frac{5}{7}} - 1]$$

$$= 0.169$$

The required probability is

$$0.35 \times 0.17 = 0.06$$