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<강의자료 Teacher's Corner>

Applications of the Continuous Uniform Distribution : Further Remarks

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1. Country A plans to invade Country B. B just signed an international treaty banning the production of poison gases. However, B has a poison gas plant; it has been active for a year; it can produce two types of poison gas, X and Y; its capacity is such that $x + y \leq 10$. A has no idea whatsoever how much raw materials for the production of X and Y B could procure. Anyway, if $7 < x + y$, B can wreak havoc on A. What is the probability that A would better give up the plan?

To A, X and Y are both random variables, and its joint p.d.f $f(x, y)$ is given as

$$f(x, y) = \frac{1}{50}, \quad 0 < x < 10, \quad 0 < y < 10, \quad 0 < x + y < 10$$
$$= 0, \qquad \textit{otherwise}$$

Hence, the required probability is $P(7 < X + Y)$, which is obtained as

$$1 - \int_0^7 \int_0^{7-y} \frac{1}{50} dxdy = \frac{51}{100}$$

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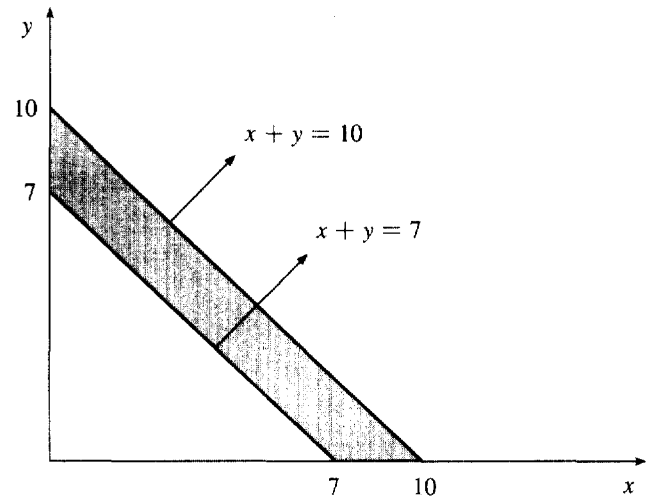
〈Figure 1〉 The Probability Corresponding to the Shaded Area



2. Suppose you intend to destroy an enemy fortress with missiles of a certain type. The type is so strong that one hit is good enough to demolish the target. You have 5 missiles and launch them in succession. Of course, there is no point in over-destruction.Unfortunately, you do not know the hit ratio. It can be anywhere between 0 and 1. How much can you be sure that you can successfully carry out the operation?

Let θ be the hit ratio. As θ is an unknown constant anywhere between 0 and 1, it can be deemed a continuous uniform random variable on a unit interval.

So,

$$f(\theta) = 1, \qquad 0 < \theta < 1$$
$$= 0, \qquad otherwise$$

Let X be the number of the missiles necessary to demolish the target. Then,

$$f(x \mid \theta) = (1-\theta)^{x-1}\theta, \qquad x = 1, 2, \cdots$$
$$= 0, \qquad otherwise$$

$$f(x) = \int_0^1 f(x, \theta)\, d\theta$$

$$= \int_0^1 f(x \mid \theta) f(\theta)\, d\theta$$

$$= \int_0^1 (1-\theta)^{x-1}\theta \cdot 1 d\theta$$

$$= (x-1)!\, \frac{1}{(x+1)!} \int_0^1 \frac{(x+1)!}{(x-1)!1!}(1-\theta)^{x-1}\theta\, d\theta$$

$$= \frac{1}{x(x+1)}$$

It is easy to verify $\sum_{x=1}^{\infty} \frac{1}{x(x+1)} = 1.$[1)]

$$P(X \le 5) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6}$$

$$= 0.8333$$

An Alternative Model

Suppose 5 hits are necessary to demolish the enemy fortress and the hit ratio is totally unknown. Then,

$$f(x) = \int_0^1 f(x \mid \theta) f(\theta)\, d\theta$$

$$= \int_0^1 \binom{4+x}{x} \theta^4 (1-\theta)^x \theta 1 d\theta$$

$$= \binom{4+x}{x} \frac{5!\, x!}{(6+x)!}$$

$$= \frac{5}{(6+x)(5+x)}, \qquad x = 0, 1, 2, \cdots$$

$$= 0, \qquad \textit{otherwise}$$

$$5 \sum_{x=0}^{\infty} \frac{1}{(6+x)(5+x)}$$

$$= 5\left(1 - \frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5}\right)$$

$$= 5 \cdot \frac{1}{5} = 1$$

1) Sokolnikoff-Redheffer, *Mathematics of Physics and Modern Engineering*, 2nd edition, 1960, p. 3.

3. A man by the name of $\mathscr{L}$ opens a grocery in a residential area. He intends to sell two brands of beer, A and B. However, he has no idea whatsoever about the market share of A. Under these conditions, what will be the probability that $\mathscr{L}$ sells more than 80 bottles of A when the total sale is 100 bottles?

Let θ be the market share of A. As θ is an unknown constant and no information is available on it, it may be considered to take any value between 0 and 1. So, θ should be deemed a continuous uniform random variable on a unit interval.

$$\begin{aligned} f(\theta) &= 1, \qquad 0 < \theta < 1 \\ &= 0, \qquad \textit{otherwise} \end{aligned}$$

On the other hand : Let X be the number of A bottles; then,

$$\begin{aligned} f(x \mid \theta) &= \binom{100}{x} \theta^x (1-\theta)^{100-x}, \qquad x = 0, 1, 2, \cdots, 100 \\ &= 0, \qquad \textit{otherwise} \end{aligned}$$

$$\begin{aligned} f(x) &= \int_0^1 f(x \mid \theta) f(\theta)\, d\theta \\ &= \int_0^1 \binom{100}{x} \theta^x (1-\theta)^{100-x} \cdot 1 d\theta \\ &= \frac{1}{100+1} \int_0^1 \frac{101!}{x!(100-x)!} \theta^x (1-\theta)^{100-x} d\theta \\ &= \begin{cases} \frac{1}{100}, & x = 0, 1, 2, \cdots, 100 \\ 0, & \textit{otherwise} \end{cases} \end{aligned}$$

The outcome well reflects our intuition that any number between 0 and 100, inclusive, should be equally possible when no information on θ, the market share of A, is available.

The required probability, $P(X \geq 80) = \frac{21}{101}$.

An Alternative Model

Suppose there is no ground to assume that either A or B is favored over the other, and hence that though θ, the market share of A, can take any value between 0 and 1, it can most likely take the value of $\frac{1}{2}$. Then, the simplest Beta-function,

$$f(\theta) = \frac{3!}{1!1!}\theta^1(1-\theta)^1, \quad 0<\theta<1$$
$$= 0, \qquad otherwise$$

describes the situation.

Let n be the number of total bottles sold, and X be the number of A's sold.

$$f(x) = \int_0^1 f(x\mid\theta)f(\theta)\,d\theta$$
$$= \int_0^1 \binom{n}{x}\theta^x(1-\theta)^{n-x}\cdot\frac{3!}{1!1!}\theta^1(1-\theta)^1 d\theta$$
$$= \frac{6(n-x+1)(x+1)}{(n+3)(n+2)(n+1)}, \quad x = 0, 1, 2, \cdots, n$$

$$\sum_{x=0}^{n}(n-x+1)(x+1)$$
$$= \frac{1}{2}\cdot\frac{1}{3}(n+2)((n+2)^2-1)$$
$$= \frac{1}{6}(n+1)(n+2)(n+3)$$

4. Machine I is put up for bidding. Its estimated net future income, inclusive of depreciation, is \$1,000,000, and its estimated manufacturing cost is \$800,000. So, bidding conditions are: No collusion is tolerated; the minimum bidding price is \$800,000. Under these conditions, the number of bidders will be a Poisson random variable with parameter 5. What is the probability that the machine will fetch more than \$900,000?

Converting \$100,000 into 1K, let $X_i - 8$, $i = 1, 2, \cdots, n$. be the ith bidder's bidding price. Then,

$$f(x_i) = \frac{1}{2}, \quad 0 < x_i < 2$$
$$= 0, \qquad otherwise$$

Let n be the number of bidders. Then,

$$f(n) = \frac{e^{-5}\cdot 5^n}{n!} \Big/ \sum_{n=2}^{\infty} \frac{e^{-5}\cdot 5^n}{n!}, \qquad n = 2, 3, \cdots$$
$$= 0, \qquad otherwise$$

Let $U = \max\{X_1, X_2, \cdots, X_n\}$. Then,

$$f(u \mid n) = nu^{n-1}\left(\frac{1}{2}\right)^n, \qquad 0 < u < 2$$
$$= 0, \qquad otherwise$$

$$f(u, n) = f(u \mid n) f(n)$$
$$= nu^{n-1}\left(\frac{1}{2}\right)^n \cdot \frac{\frac{e^{-5}\cdot 5^n}{n!}}{1-e^{-5}-e^{-5}\cdot 5}$$

$$f(u) = \sum_{n=2}^{\infty} f(u, n)$$
$$= \frac{1}{2}e^{-5}\cdot 5 \cdot e^{\frac{5u}{2}} \sum_{n=2}^{\infty} \frac{e^{\frac{-5u}{2}}\left(\frac{5u}{2}\right)^{n-1}}{(n-1)!} \Big/ (1-e^{-5}-e^{-5}\cdot 5)$$
$$= \frac{1}{2}e^{-5}\cdot 5 \cdot e^{\frac{5u}{2}} \frac{1-e^{\frac{-5u}{2}}}{1-e^{-5}-e^{-5}\cdot 5}$$

$$\int_0^2 f(u)\, du = \int_0^2 \frac{\frac{1}{2}e^{-5}\cdot 5\cdot e^{\frac{5u}{2}}\left(1-e^{\frac{-5u}{2}}\right)}{1-e^{-5}-e^{-5}\cdot 5}\, du$$

The required probability,

$$P(U > 1) = \int_1^2 \frac{1}{2} 5 \cdot e^{-5} e^{\frac{5u}{2}} \frac{1-e^{\frac{-5u}{2}}}{1-e^{-5}-5e^{-5}}\, du$$
$$= \frac{1-e^{-2.5}-2.5e^{-5}}{1-e^{-5}-5e^{-5}}$$

5. Intelligence agent α of Country A is superbly active in enemy Country B. However, his identity is finally revealed, and he is being chased. Driving a Type P car with the maximum speed of 60 MPH on a sparsely travelled highway, he passes Spot K and proceeds to an unguarded national border 60 miles off K; there is no regular road beyond the national border. A squad of counter intelligence agents of B, moving in a Type Q car with the maximum speed of 90 MPH, belatedly arrives at K 10 minutes after α passes. The squad immediately learns about α's movement and supposes: α knows that he is the most wanted man in B, but does not know anything about the squad's pursuit; α may abandon his car anywhere between K and the border and disappear in the thick forest lying along the highway. What is the probability that the squad catches α on the highway from the squad's viewpoint?

Let X be α's car-abandon time in minutes, starting at K. Then,

$$\begin{aligned} f(x) &= \frac{1}{60}, \quad 0 < x < 60 \\ &= 0, \quad \textit{otherwise} \end{aligned}$$

In order to begin the chase, the squad must assume that α is still driving when it arrives at K. If α abandons his car and disappears in to the forest within 20 minutes, the squad's chasing race will be futile.

The required probability,

$$\begin{aligned} P((X > 20 \mid X > 10) &= \frac{P(X > 30)}{P(X > 10)} \\ &= \frac{\frac{30}{60}}{\frac{50}{60}} = \frac{3}{5} \end{aligned}$$

6. Firm A needs 1,000 units of a certain type of good. A can procure the required amount from domestic Producers I or II or from abroad. Producer II has no idea how A will apportion the requirement. What is the probability, from II 's view point, that II will get an order of at least 200 units?

Let A apportions x units to I and y units to II .

Then,

$$f(x, y) = \frac{1}{500{,}000}, \quad 0 < x < 1{,}000, \quad 0 < y < 1{,}000$$
$$0 < x + y < 1{,}000$$
$$= 0, \quad otherwise$$

$$f(y) = \int_0^{1{,}000-y} \frac{1}{500{,}000}\, dx$$
$$= \frac{1}{500{,}000}(1{,}000-y), \quad 0 < y < 1{,}000$$
$$= 0, \quad otherwise$$

The required probability,

$$P(Y > 200) = \int_{200}^{1{,}000} \frac{1}{500{,}000}(1{,}000-y)\, dy = 0.64$$

7. Country *A* is in war with Country *B*. *A* has an explosive powder plant inland. *B*'s special task force knows that the plant's monthly capacity is 100 units, but has no idea on how much will have been produced at the plant this month and how much will have been transferred to a newly built depot on a seashore. If the depot will have received more than 50 units, the task force will attempt to demolish the depot at the beginning of next month. Should the task force launch an operation?

Let X be the amount produced at the plant, and Y be the amount transferred to the depot. Then,

$$f(x, y) = \frac{1}{5{,}000}, \quad 0 < x < 100, \quad 0 < y < x$$
$$= 0, \quad otherwise$$

$$f(y) = \int_0^{100} \frac{1}{5{,}000}\, dx = \frac{1}{5{,}000}(100-y), \quad 0 < y < 100$$
$$= 0, \quad otherwise$$

The probability that more than 50 units will have been transferred to the depot,

$$P(Y > 50) = \int_{50}^{100} \frac{1}{5{,}000}(100-y)\, dy = 0.25$$