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〈강의자료 Teacher's Corner〉

Applications of the Continuous Uniform Distribution

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Though the simplest among continuous probability density functions, the continuous uniform distribution admits interesting applications.

1. Suppose a government invites n construction firms to join in bidding for a governmental building project ; the government independently estimates that the minimum construction cost will be \$5,000 per square meter, and procures a budget of \$7,000 per square meter. So, the government declares : No collusion will be tolerated ; the maximum bidding price is \$7,000 per square meter. The government may assume a bidder's price will be equally possible anywhere between \$5,000 and \$7,000. The question is : On what grounds may the government be suspicious of collusion?

A continuous uniform distribution serves the purpose. Let $x+5{,}000$ be the bidder's price. Then

$$f(x) = \frac{1}{2{,}000}, \qquad 0 < x < 2{,}000$$
$$= 0, \qquad otherwise$$

Let $\{x_1, x_2, \cdots, x_n\}$ be bidder's prices, and $u = \min\{x_1, x_2, \cdots, x_n\}$. Then, the cumulative distribution function of U,

$$H(u) = 1 - \left(1 - \frac{u}{2{,}000}\right)^n$$

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and the probability density function of U,

$$h(u) = n\left(1 - \frac{u}{2,000}\right)^{n-1} \cdot \frac{1}{2,000}$$

If $n = 5$, and $u = 1,800$, $P(1,800 \leq U) = \frac{1}{10,000}$, a good ground to be suspicious of collusion.

2. A team of 6 professionals on a mission impossible is dispatched into an enemy's territory. Each of them should separately arrive at a spot between 0 : 00 and 1 : 00 on a certain day ; none of them can wait for the others for more than 10 minutes, so all the arrivals should be completed within 10 minutes. How likely can the team launch the operation?

The question reduces to finding the probability density function for the range of a sample of size 6 from a population with density

$$f(x) = \frac{1}{60}, \quad 0 < x < 60$$
$$= 0, \quad \textit{otherwise}$$

Let $x_1, x_2, x_3, x_4, x_5, x_6$ be the arrival times of the 6 professionals, respectively, and y_1, y_2, y_3, y_4, y_5, y_6 be the ordered x_i's from the smallest to the largest.

$$\text{Let } u = y_6 - y_1, \quad \text{Then } f(u) = 30\,u^4\,(60 - u)\frac{1}{60^6}, \quad 0 < u < 60$$
$$= 0, \quad \textit{otherwise}$$ [1]

The answer :

$$\int_0^{10} 30\,u^4\,(60 - u)\,\frac{1}{60^6}\,du = \left(\frac{1}{6}\right)^6 31$$

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1) Mood, A. M. and F. A. Graybill, *Introduction to the theory of statistics*, second edition, New York; McGraw-Hill, 1963, pp. 240~242.

3. In the bidding of 1., the government officials very much favor bidder i, so they secretly decide among themselves : if none of the remaining bidders bids lower than bidder i by more than \$ 100, then he will have the project. Will favoritism take effect?

Let $s = \min\{x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n\}$. Then,

the cumulative distribution function of s,

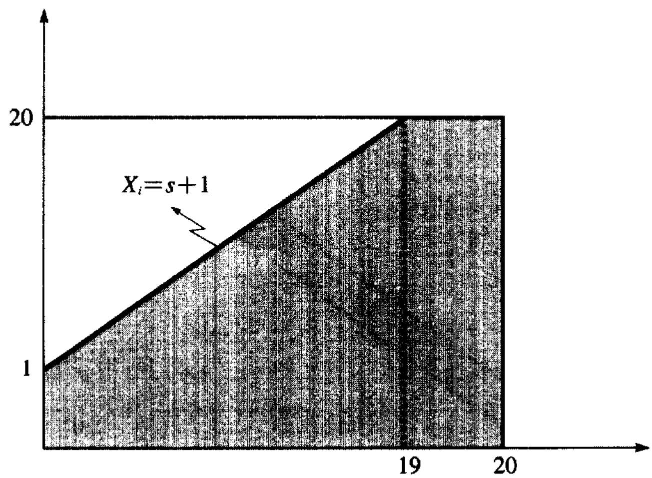
$$F(s) = 1 - P(S > s) = 1 - \left(1 - \frac{s}{2{,}000}\right)^{n-1},$$

and the probability density function of s,

$$f(s) = (n-1)\left(1 - \frac{s}{2{,}000}\right)^{n-2} \frac{1}{2{,}000}, \qquad 0 < s < 2{,}000$$
$$= 0, \qquad \textit{otherwise}$$

$$P(X_i - 100 < S) = P(X_i < S + 100)$$
$$= \int_0^{1{,}900}\int_0^{s+100} \frac{1}{2{,}000}(n-1)\left(1-\frac{s}{2{,}000}\right)^{n-2}\frac{1}{2{,}000}\,dxds + \int_{1{,}900}^{2{,}000}\int_0^{2{,}000} \frac{1}{2{,}000}(n-1)\left(1-\frac{s}{2{,}000}\right)^{n-2}\frac{1}{2{,}000}\,dxds$$

〈Figure 1〉 (S, X_i) satisfying the condition $X_i < S + 100$ should be in the shaded area



$$=\frac{1}{20}+\frac{1}{n}-\frac{1}{n}\left(\frac{1}{20}\right)^{n}$$

As $n \longrightarrow \infty$, i will obtain the project with probability, $\frac{1}{20}$.

4. Firm A needs a certain kind of machine. Manufactures B and C produce the same type ; However, B's is a little bit superior to C's. A estimates that the production cost of both B and C is K100, and wants to buy the machine at less than K120. The business world is bizzare : A knows that C's man is planted in B, so B's pricing strategy will leak to C. A invites B and C for bidding with the proviso that the maximum bidding price should be K120, and the quality difference will be taken into consideration.

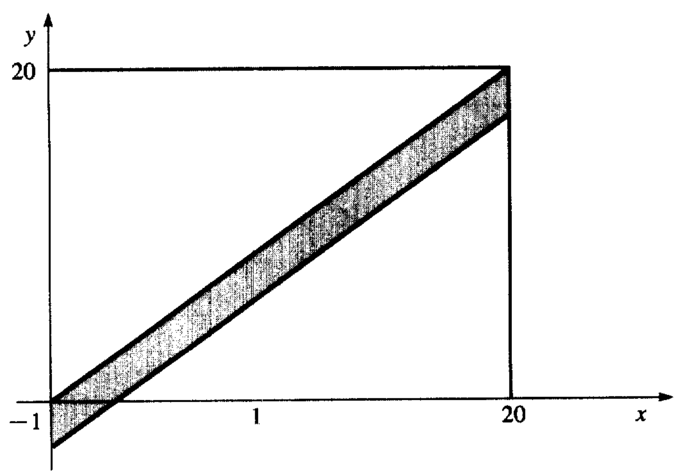
Let $X+100$ be B's bidding price, and $Y+100$ be C's. Then

$$f(x, y)=\frac{1}{10}, \qquad 0<y<x, \qquad 0<x<20$$
$$=0, \qquad \textit{otherwise}$$

Though X and Y are not statistically independent, f is certainly a bivariate continuous uniform probability density function.

A decides : if B's bidding price does not exceed C's by more than K1, B will have the order. What is the probability that B outbids C from A's viewpoint? The answer :

〈Figure 2〉 (x, y) satisfying the condition $x-1<y<x$ should be in the shaded area



$$P(X-Y<1)=P(Y>X-1)$$
$$=1-\int_1^{20}\int_0^{x-1}\frac{1}{200}\,dydx=\frac{19.5}{200}$$

5. Suppose a value that a continuous random variable X can take is bounded and no information on the "randomness" of X is available ; then, we are forced to postulate :

$$f(x)=\frac{1}{\beta-\alpha},\qquad \alpha<x<\beta$$
$$=0,\qquad \textit{otherwise}$$

When a fixed amount of a newly bred seed X is planted on an unaccustomed plot or when a drug dealer launches his business in a virgin territory, the yields or the sales should be assumed continuous uniform random variables.

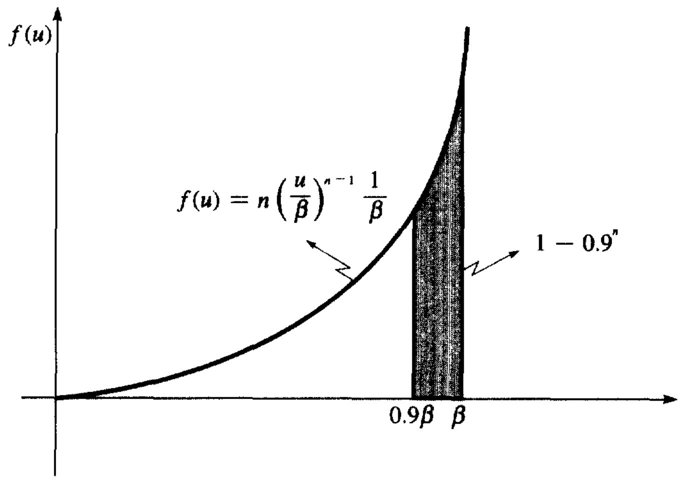
Suppose α is known to be 0. Then,

$$f(x)=\frac{1}{\beta},\qquad 0<x<\beta$$
$$=0,\qquad \textit{otherwise}$$

The maximum likelihood estimator for β is $u=\max\{x_1, x_2, x_3,\cdots, x_n\}$, where $\{x_1, x_2, x_3,\cdots, x_n\}$ is a random sample from the population, and

$$f(u)=n\left(\frac{u}{\beta}\right)^{n-1}\cdot\frac{1}{\beta}$$
$$E(U)=\mu_U=\frac{n}{n+1}\beta$$
$$E(U-\mu_U)^2=\sigma^2_U=\frac{n}{(n+1)^2(n+2)}\beta^2$$

The interval estimation of β has not attracted much attention. The task is straightforward. A 95% confidence interval of β based on U is as follows :



If u falls in $(0.9\beta, \beta)$, then $\left(u, \frac{u}{0.9}\right)$ covers β with confidence coefficient 95%, where $u = 0.9\beta$ and hence $\beta = \frac{u}{0.9}$. Suppose u falls in $(0.9\beta, 0.9\beta + \varepsilon)$. Then, $\left(u, \frac{u}{0.9}\right)$ certainly covers β.

Let $\int_{\hat{u}}^{\beta} n\left(\frac{u}{\beta}\right)\frac{1}{\beta}du = 0.95$. Then, $\hat{u} = 0.05^{\frac{1}{n}}\ \beta$, and $\beta = \frac{\hat{u}}{0.05^{\frac{1}{n}}}$.

$\left(\hat{u}, \frac{\hat{u}}{0.05^{\frac{1}{n}}}\right)$ covers β with confidence coefficient, 95%.

For $n = 16$, $(\hat{u}, 1.2059\ \hat{u})$; for $n = 32$, $(\hat{u}, 1.0981\ \hat{u})$.