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Observation Contemporaneity and Estimation Efficiency in the Context of Identical Regressors

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For a simple two-equation case of the Seemingly Unrelated Regression(SUR) model, Im [2] shows that there is a partial efficiency gain from Generalized Lest Squares(GLS) estimation in the context of both identical regressors and unequal numbers of observations. This note fully generalizes the limited context of the earlier work, showing that there is a global, partial, or no efficiency gain from GLS (estimation), depending on the structure of the observation contemporaneity.

I. Introduction

In the context of identical regressors in the SUR model introduced by Zellner [3], Dwivedi and Srivastava [1] show that there is no efficiency gain from GLS. Later, Im [2] discusses the partial efficiency gain from GLS in the context of both identical regressors and unequal numbers of observations for two equation case. Therein, we deal with a case of asymmetric sample observation contemporaneity between the two equations: all sample observations for the first equation have contemporaneous counterparts in the sample for the second, while the converse is not the case, and it has been shown that there is an efficiency gain from GLS for the first equation and no efficiency gain for the second.

Herein, we consider the efficiency gain in a general context, general both with

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respect to the number of equations and observation contemporaneity. We find that there is either global, partial, or no-depending on how observation contemporaneity configures between equations-efficiency gain from GLS.

To faciliate an analytical shortcut without loss of generality, we make the following assumptions on the standard SUR model :

Assumption 1 There are $m(=a+b)$ equations with k identical regressors, a of which have T_A $(=p+q)$ observations, and b of which T_B $(=q+r)$ observations, where integers a, b, $k \geq 1$ and p, q, $r \geq k$ (The a and b equations henceforth will be referred to as (equation group) A and (equation group) B, respectively).

Assumption 2 All equations within each equation group have equal numbers of observations, all contemporaneous.

Assumption 3 q observations for A are contemporaneous with the equal number of observations for B, while the remaining p observations for A are not contemporaneous with any of the remaining q observations for B (The p, q and r observations henceforth will be referred to as observation group P, Q and R, respectively, or simply P, Q and R).

Ⅱ. General Case

Reflecting the assumptions in Section Ⅰ, we can write each equation group for each of its observation groups with subscripts A, B, P, Q and R signifying equation groups A and B, and observation groups P, Q and R, respectively :

$$\boldsymbol{Y}_{A.P} = (\boldsymbol{I}_a \otimes \boldsymbol{X}_P)\boldsymbol{\beta}_A + \boldsymbol{U}_{A.P}, \qquad \boldsymbol{Y}_{A.Q} = (\boldsymbol{I}_a \otimes \boldsymbol{X}_Q)\boldsymbol{\beta}_A + \boldsymbol{U}_{A.Q} \tag{1}$$

and

$$\boldsymbol{Y}_{B.Q} = (\boldsymbol{I}_b \otimes \boldsymbol{X}_Q)\boldsymbol{\beta}_B + \boldsymbol{U}_{B.Q}, \qquad \boldsymbol{Y}_{B.R} = (\boldsymbol{I}_b \otimes \boldsymbol{X}_R)\boldsymbol{\beta}_B + \boldsymbol{U}_{B.R} \tag{2}$$

where the matrix dimensions are :

$$\boldsymbol{Y}_{A.P} = ap \times 1, \quad \boldsymbol{Y}_{A.Q} = aq \times 1, \quad \boldsymbol{Y}_{B.Q} = bq \times 1, \quad \boldsymbol{Y}_{B.R} = br \times 1$$
$$\boldsymbol{X}_P = p \times k, \quad \boldsymbol{X}_Q = q \times k, \quad \boldsymbol{X}_R = r \times k$$
$$\boldsymbol{\beta}_A = ak \times 1, \quad \boldsymbol{\beta}_B = bk \times 1$$
$$\boldsymbol{U}_{A.P} = ap \times 1, \quad \boldsymbol{U}_{A.Q} = aq \times 1, \quad \boldsymbol{U}_{B.P} = bp \times 1, \quad \boldsymbol{U}_{B.Q} = bq \times 1$$

Define

$$\sum : = \begin{bmatrix} \sum_{AA} & \sum_{AB} \\ \sum_{BA} & \sum_{BB} \end{bmatrix} : = \begin{bmatrix} E(\boldsymbol{U}_{A,}\boldsymbol{U}'_{A,}) & E(\boldsymbol{U}_{A,}\boldsymbol{U}'_{B,}) \\ E(\boldsymbol{U}_{B,}\boldsymbol{U}'_{A,}) & E(\boldsymbol{U}_{B,}\boldsymbol{U}'_{B,}) \end{bmatrix} \tag{3}$$
$$\sum^{-1} : = \begin{bmatrix} \sum^{AA} & \sum^{AB} \\ \sum^{BA} & \sum^{BB} \end{bmatrix}$$

where partitions in $\sum$ and $\sum^{-1}$ corresponds in dimensions :

$$\sum_{AA}, \sum^{AA} = a \times a, \qquad \sum_{AB}, \sum^{AB} = a \times b$$
$$\sum_{BA}, \sum^{BA} = b \times a, \qquad \sum_{BB}, \sum^{BB} = b \times b$$

Then, from the partitioned inverse of $\sum^{-1}$, we obtain :

$$\sum_{AA} = (\sum^{AA} - \sum^{AB}(\sum^{BB})^{-1}\sum^{BA})^{-1} \tag{4}$$
$$\sum_{BB} = (\sum^{BB} - \sum^{BA}(\sum^{AA})^{-1}\sum^{AB})^{-1}$$

Now, let equation group specific or local GLS estimators of $\boldsymbol{\beta}_A$ in (1) and $\boldsymbol{\beta}_B$ in (2) by $\tilde{\boldsymbol{\beta}}_A$ and $\tilde{\boldsymbol{\beta}}_B$, respectively, and OLS estimators by $\boldsymbol{b}_A$ and $\boldsymbol{b}_B$. Then, since the regressors are identical and the observations within each equation group are contemporaneous as assumed,

$$Cov(\tilde{\boldsymbol{\beta}}_A) = \sum_{AA} \otimes (\boldsymbol{X}'_P\boldsymbol{X}_P + \boldsymbol{X}'_Q\boldsymbol{X}_Q)^{-1} \tag{5}$$
$$= Cov(\boldsymbol{b}_A)$$
$$Cov(\tilde{\boldsymbol{\beta}}_B) = \sum_{BB} \otimes (\boldsymbol{X}'_Q\boldsymbol{X}_Q + \boldsymbol{X}'_R\boldsymbol{X}_R)^{-1} \tag{6}$$
$$= Cov(\boldsymbol{b}_B)$$

in light of Dwivedi and Srivastava [1].

Stacking (1) and (2) for system wide or global GLS of $\boldsymbol{\beta}_A$ and $\boldsymbol{\beta}_B$, with '0' henceforth denoting a null matrix of an appropriate order,

$$\begin{bmatrix} Y_{A.P} \\ Y_{A.Q} \\ Y_{B.Q} \\ Y_{B.R} \end{bmatrix} = \begin{bmatrix} I_a \otimes X_P & 0 \\ I_a \otimes X_Q & 0 \\ 0 & I_b \otimes X_Q \\ 0 & I_b \otimes X_R \end{bmatrix} \begin{bmatrix} \beta_A \\ \beta_B \end{bmatrix} + \begin{bmatrix} U_{A.P} \\ U_{A.Q} \\ U_{B.Q} \\ U_{B.R} \end{bmatrix} \tag{7}$$

or compactly

$$Y = W\beta + U \tag{8}$$

Then,

$$\Omega = E(UU') = \begin{bmatrix} \Sigma_{AA} \otimes I_p & 0 & 0 \\ 0 & \Sigma \otimes I_q & 0 \\ 0 & 0 & \Sigma_{BB} \otimes I_r \end{bmatrix} \tag{9}$$

Denoting the global GLS estimator of β in (8) by $\hat{\beta}$,

$$Cov(\hat{\beta}) = [W'\Omega^{-1}W]^{-1}$$

or in partitioned form :

$$\begin{bmatrix} Cov(\hat{\beta}_A) & Cov(\hat{\beta}_A, \hat{\beta}_B) \\ Cov(\hat{\beta}_B, \hat{\beta}_A) & Cov(\hat{\beta}_B) \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} \Sigma_{AA}^{-1} \otimes X'_P X_P + \Sigma^{AA} \otimes X'_Q X_Q & \Sigma^{AB} \otimes X'_Q X_Q \\ \Sigma^{BA} \otimes X'_Q X_Q & \Sigma^{BB} \otimes X'_Q X_Q + \Sigma_{BB}^{-1} \otimes X'_R X_R \end{bmatrix}^{-1}$$

Working out the partitioned inverse of (10),

$$\begin{aligned} & Cov(\hat{\beta}_A) \\ &= [\Sigma_{AA}^{-1} \otimes X'_P X_P + \Sigma^{AA} \otimes X'_Q X_Q - (\Sigma^{AB} \otimes X'_Q X_Q)(\Sigma^{BB} \otimes X'_Q X_Q + \Sigma_{BB}^{-1} \otimes X'_R X_R)^{-1}(\Sigma^{BA} \otimes X'_Q X_Q)]^{-1} \\ &\leq [\Sigma_{AA}^{-1} \otimes X'_P X_P + \Sigma^{AA} \otimes X'_Q X_Q - (\Sigma^{AB} \otimes X'_Q X_Q)(\Sigma^{BB} \otimes X'_Q X_Q)^{-1}(\Sigma^{BA} \otimes X'_Q X_Q)]^{-1} \\ &= [\Sigma_{AA}^{-1} \otimes X'_P X_P + (\Sigma^{AA} - \Sigma^{AB}(\Sigma^{BB})^{-1}\Sigma^{BA}) \otimes X'_Q X_Q)]^{-1} \\ &= [\Sigma_{AA}^{-1} \otimes (X'_P X_P + X'_Q X_Q)]^{-1} \\ &= \Sigma_{AA} \otimes (X'_P X_P + X'_Q X_Q)^{-1} \\ &= Cov(\tilde{\beta}_A) = Cov(b_A) \end{aligned} \tag{11}$$

$$\begin{aligned} & Cov(\hat{\beta}_B) \\ &= [\Sigma^{BB} \otimes X'_Q X_Q + \Sigma_{BB}^{-1} \otimes X'_R X_R - (\Sigma^{BA} \otimes X'_Q X_Q)(\Sigma_{AA}^{-1} \otimes X'_P X_P + \Sigma^{AA} \otimes X'_Q X_Q)^{-1}(\Sigma^{AB} \otimes X'_Q X_Q)]^{-1} \end{aligned}$$

$$\leq [\Sigma^{BB} \otimes X'_Q X_Q + \Sigma_{BB}^{-1} \otimes X'_R X_R - (\Sigma^{BA} \otimes X'_Q X_Q)(\Sigma^{AA} \otimes X'_Q X_Q)^{-1}(\Sigma^{AB} \otimes X'_Q X_Q)]^{-1}$$
$$= [\Sigma_{BB}^{-1} \otimes X'_R X_R + (\Sigma^{BB} - \Sigma^{BA}(\Sigma^{AA})^{-1}\Sigma^{AB}) \otimes X'_Q X_Q]^{-1}$$
$$= [\Sigma_{BB}^{-1} \otimes (X'_R X_R + X'_Q X_Q)]^{-1}$$
$$= \Sigma_{BB} \otimes (X'_Q X_Q + X'_R X_R)^{-1}$$
$$= Cov(\tilde{\beta}_B) = Cov(b_B) \tag{12}$$

noting $\Sigma^{BB} \otimes X'_Q X_Q + \Sigma_{BB}^{-1} X'_R X_R \geq \Sigma^{BB} \otimes X'_Q X_Q$ and $\Sigma_{AA}^{-1} \otimes X'_P X_P + \Sigma^{AA} \otimes X'_Q X_Q \geq \Sigma^{AA} \otimes X'_Q X_Q$, and reflecting (4), (5) and (6).

In summary of (11) and (12),

$$Cov(\hat{\beta}_A) \leq Cov(b_A), \qquad Cov(\hat{\beta}_B) \leq Cov(b_B) \tag{13}$$

i.e., the global GLS results in semidefinite efficiency gains for both equation groups, A and B(Efficiency gains are definite if $X'_Q X_Q$ is positive definite).

Ⅲ. Special Cases

Suppose that observation group P for equation group A in (1) (and subsequently (7)), is non-existent(i.e., unavailable). Then, X_P products will not appear in (5), (10), (11) and (12). Consequently, from (11) and (12) we easily get ;

$$Cov(\hat{\beta}_A) \leq Cov(b_A), \qquad Cov(\hat{\beta}_B) \leq Cov(b_B) \tag{14}$$

i.e., only the equation group with smaller number of observations has efficiency gain (The case dealt with in Im [2] is a two-equation example of (14)).

If instead the observation group R for equation group B in (2) is non-existent, then X_R products will not appear in (6), (10), (11) and (12) so that we get the reverse of (14) ;

$$Cov(\hat{\beta}_A) = Cov(b_A), \qquad Cov(\hat{\beta}_B) \leq Cov(b_B) \tag{15}$$

If both observation groups P and R are non-existent in (1) and (2), then we have a perfect case of Dwivedi and Srivastava [1], therefore no efficiency gain from GLS. If observation group Q is non-existent in (1) and (2), we have a trivial case where the contemporaneous covariance matrix is irrelevant, hence no efficiency gain from GLS. In either case, similarly to the way (14) and (15) are arrived at, we can easily confirm the no efficiency gain from (11) and (12) ;

$$Cov(\hat{\beta}_A) = Cov(b_A), \qquad Cov(\hat{\beta}_B) = Cov(b_B) \tag{16}$$

Ⅳ. Concluding Remarks

This note generalizes the exploratory work by Im [2], which deals with a simple two-equation case of asymmetric observation contemporaneity, with respect to the number of equations and observation contemporaneity. Unlike a plausible intuitive inclination to no efficiency gain from GLS in light of Dwivedi and Srivastava [1], we find that even in the context of identical regressors there is a global, partial, or no-depending on the configuration of the observation contemporaneity between equation groups-efficiency gain from GLS.

We have partitioned the observations and equations into three and two groups, respectively, with no loss of generality : with more refined partitioning, one can only add to the algebraic burden without no additional point there is to make. Neverthless, the burden would not be overwhelming with a large but limited number of partitions since the disturbance covariance matrix will remain block diagonal regardless of the number of partitions.

❖ *REFERENCES* ❖

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