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The Asymptotic Equivalence of the Iterative Least Squares Estimators and the Extended Yule-Walker Estimators of the AR Parameters for Stationary ARMA Models

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The iterative least squares procedure was proposed to estimate the autoregressive moving-average processes. The consistency of the estimates was also shown. In this paper it will be shown that in case of stationary processes the iterative least squares estimates are asymptotically equivalent to the extended Yule-Walker estimates of the autoregressive parameters, i.e., the extended Yule-Walker estimates are consistent, and have the same asymptotic distribution as the iterative least squares estimates.

I. Introduction

Consider the stationary ARMA(p, q) model for a univariate time series $\{Z_t\}$,

$$\phi(B)Z_t = \theta(B)a_t, \qquad (1)$$

where $\phi(x) = -\phi_0 - \phi_1 x - \cdots - \phi_p x^p$, and $\theta(x) = -\theta_0 - \theta_1 x - \cdots - \theta_q x^q$ are polynomials in x ; $\phi_0 = \theta_0 = -1$; $\{a_t\}$ is a white noise process with variance σ^2; and B is the backshift operator. We assume that all the zeros of both $\phi(x)$ and $\theta(x)$ are outside the unit circle, and also assume that $\phi(x)$ and $\theta(x)$ have no common zeros. We denote the autocovariance function (ACVF) and the autocorrelation function (ACRF) of the process by $\{\sigma(j)\}$ and $\{\rho_j\}$, respectively. Also, let $\{r_j\}$ be the sample ACRF.

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Tsay and Tiao [3] proposed an iterative regression procedure that yields consistent least squares (LS) estimates for the autoregressive parameters of the model (1). The purpose of this paper is to show that in case of stationary processes the iterative LS estimates are asymptotically equivalent to the extended Yule-Walker (EYW) estimates.

II. The Iterative LS Estimates

Suppose n observations are available from the stationary ARMA(p, q) model (1). The iterative regression procedure can be summarized as follows. For each $k = 1, 2, \cdots$, we first fit the observations to an AR(k) model,

$$Z_t = \sum_{m=1}^{k} \phi_{m(k)}^{(0)} Z_{t-m} + e_{k,t}^{(0)}, \qquad t = k+1, \cdots, n.$$

The estimated residuals are

$$\hat{e}_{k,t}^{(0)} = Z_t - \sum_{m=1}^{k} \hat{\phi}_{m(k)}^{(0)} Z_{t-m}.$$

Next, we regress the data to the first iterated AR(k) model as

$$Z_t = \sum_{m=1}^{k} \phi_{m(k)}^{(1)} Z_{t-m} + \beta_{1(k)}^{(1)} \hat{e}_{k,t-1}^{(0)} + e_{k,t}^{(1)}, \qquad t = k+2, \cdots, n.$$

Then, the estimated residuals are

$$\hat{e}_{k,t}^{(1)} = Z_t - \sum_{m=1}^{k} \hat{\phi}_{m(k)}^{(1)} Z_{t-m} - \hat{\beta}_{1(k)}^{(1)} \hat{e}_{k,t-1}^{(0)}.$$

Similarly, the j-th iterated AR(k) regression of $\{Z_t\}$ is defined as, for j=2, 3, $\cdots$,

$$Z_t = \sum_{m=1}^{k} \phi_{m(k)}^{(j)} Z_{t-m} + \sum_{i=1}^{j} \beta_{i(k)}^{(j)} \hat{e}_{k,t-i}^{(j-i)} + e_{k,t}^{(j)}, \qquad t = k+j+1, \cdots, n.$$

Here,

$$\hat{e}_{k,t}^{(j)} = Z_t - \sum_{m=1}^{k} \hat{\phi}_{m(k)}^{(j)} Z_{t-m} - \sum_{h=1}^{j} \hat{\beta}_{h(k)}^{(j)} \hat{e}_{k,t-h}^{(j-h)}$$

is the estimated residual of the j-th iterated AR(k) regression, and $\hat{\phi}_{m(k)}^{(j)}$ and $\hat{\beta}_{h(k)}^{(j)}$ are the corresponding LS estimates.

Equation (2.7) of [3] is a useful formula to calculate all the iterative LS estimates recursively as follows. For initial estimation, the ordinary least squares (OLS) method is used to obtain $\hat{\phi}^{(0)}_{1(k)}, \cdots, \hat{\phi}^{(0)}_{k(k)}$ for each $k = 1, 2, \cdots$. It is equivalent to solving the normal equations

$$\sum_t \hat{e}^{(0)}_{k,t} Z_{t-h} = \sum_t \{Z_t - \sum_{m=1}^{k} \hat{\phi}^{(0)}_{m(k)} Z_{t-m}\} Z_{t-h} = 0, \qquad h = 1, \cdots, k. \quad (2)$$

Equations (2) becomes

$$r_h = \sum_{m=1}^{k} \hat{\phi}^{(0)}_{m(k)} r_{h-m} + O_P(n^{-1}), \qquad h = 1, \cdots, k. \quad (3)$$

For recursive part, the j-th iterated AR(k) regression for $k \geq 1$ and $j \geq 1$ are calculated by the algorithm,

$$\phi^{(j)}_{m(k)} = \hat{\phi}^{(j-1)}_{m(k+1)} - \hat{\phi}^{(j-1)}_{m-1(k)} \hat{\phi}^{(j-1)}_{k+1(k+1)} / \hat{\phi}^{(j-1)}_{k(k)}, \qquad m = 1, \cdots, k, \quad (4)$$

where $\hat{\phi}^{(j-1)}_{0(k)} = -1$. Since the process is not purely deterministic, all the LS estimates $\hat{\phi}^{(j)}_{m(k)}$ are nonzeros with probability one for fixed n. Thus, we can use Equation (4) without reservations.

Ⅲ. The Extended Yule-Walker Estimates

If the process is stationary, then the ACRF satisfies the extended Yule-Walker equations;

$$\rho_m = \phi_1 \rho_{m-1} + \cdots + \phi_p \rho_{m-p}, \qquad m = q+1, q+2, \cdots,$$

$$\text{i.e.,} \ \ \phi(B)\rho_m = 0, \ \ m = q+1, q+2, \cdots. \quad (5)$$

When the ACRF is given, the AR parameters $\phi_1, \cdots, \phi_p$ can be obtained by solving the EYW equations with $m = q+1, \cdots, q+p$. Usually we do not know the orders p and q. Then, it is necessary for each pair (k, j) of orders to solve the EYW equations with $m = j+1, \cdots, j+k$. Denote the solutions by $\phi^{(j)}_{k,1}, \cdots, \phi^{(j)}_{k,k}$. Then

$$\rho_m = \phi^{(j)}_{k,1} \rho_{m-1} + \cdots + \phi^{(j)}_{k,k} \rho_{m-k}, \qquad m = j+1, \cdots, j+k. \quad (6)$$

We define the extended Yule-Walker estimates $\phi^{(j)}_{k,1}, \cdots, \phi^{(j)}_{k,k}$ as the solutions of the EYW equations (6) with replacing the true ACRF with the sample ACRF.

It will be shown in Section V that the EYW estimates $\{\hat{\phi}^{(j)}_{k,m}\}$ satisfy the recursive algorithm (4), i.e., for $k \geq 1$ and $j \geq 1$

$$\hat{\phi}^{(j)}_{k,m} = \hat{\phi}^{(j-1)}_{k+1,m} - \hat{\phi}^{(j-1)}_{k,m-1}\hat{\phi}^{(j-1)}_{k+1,k+1}/\hat{\phi}^{(j-1)}_{k,k}, \qquad m = 1, \cdots, k, \tag{7}$$

where $\hat{\phi}^{(j)}_{k,0} = -1$. Moreover, the initial EYW estimates $\{\hat{\phi}^{(0)}_{k,m} \mid k = 1, 2, \cdots, \ m = 1, \cdots, k\}$ satisfy the sample Yule-Walker equations of the AR(k) model,

$$r_h = \sum_{m=1}^{k} \hat{\phi}^{(0)}_{k,m} r_{h-m}, \qquad h = 1, \cdots, k, \tag{8}$$

which is asymptotically equaivalent to (3).

Ⅳ. The Asymptotic Equivalence

Since the process is not purely deterministic, the solutions of (8) is unique for each k. For more details, refer to Lemma 2.1 of [2]. Cramer's rule implies the following lemma.

Lemma 4.1 If the underlying process is from the stationary ARMA(p,q) model (1), then

$$\hat{\phi}^{(0)}_{k,m} = \hat{\phi}^{(0)}_{m(k)} + O_p(n^{-1}), \qquad m = 1, \cdots, k.$$

One can show the following asymptotic relation of the iterative LS estimates using the same method as the proof of Lemma 6.2 of [3].

Lemma 4.2 Assume that the underlying process is from the stationary ARMA(p, q) model (1). If an estimated polynomial is defined by

$$\hat{\phi}^{(j)}_{k}(x) = -\sum_{m=0}^{k} \hat{\phi}^{(j)}_{k,m} x^m,$$

then, for $k \geq p$ and $j \geq q$;

$$\hat{\phi}^{(j)}_{k}(B) = \phi(B) H^{(j-q)}_{k-p}(B) + O_p(n^{-1/2}), \qquad 0 \leq j-q \leq k-p,$$
$$\hat{\phi}^{(j)}_{k}(B) = \phi(B) H^{(k-p)}_{j-q}(B) + O_p(n^{-1/2}), \qquad 0 \leq k-p < j-q,$$

where the $H(B)$ is the random-coefficient polynomial defined in Equation (6.16) of [3].

Similarly one can show that the consistency of $\hat{\phi}^{(j)}_{k,m}$.

Theorem 4.1 Assume that the underlying process is from the stationary ARMA(p, q) model (1). If an estimated polynomial is defined by

$$\hat{\phi}^{(j)}_{k}(x) = -\sum_{m=0}^{k} \hat{\phi}^{(j)}_{m(k)} x^{m},$$

then, for $k \geq p$ and $j \geq q$;

$$\hat{\phi}^{(j)}_{k}(B) = \phi(B) H^{(j-q)}_{k-p}(B) + O_p(n^{-1/2}), \qquad 0 \leq j-q \leq k-p,$$

$$\hat{\phi}^{(j)}_{k}(B) = \phi(B) H^{(k-p)}_{j-q}(B) + O_p(n^{-1/2}), \qquad 0 \leq k-p < j-q,$$

where the $H(B)$ is the random-coefficient polynomial defined as before.

Corollary 4.1 Under the condition of Lemma 4.2, $\hat{\phi}^{(j)}_{k+1,k+1} / \phi^{(j)}_{k,k}$ and $\hat{\phi}^{(j)}_{k+1(k+1)} / \hat{\phi}^{(j)}_{k(k)}$ are consistent to the same constant or random variable, which we will denote $\alpha^{(j)}_{k}$.

We now show the asymptotic equivalence of the iterative LS estimates and the EYW estimates for the stationary processes.

Theorem 4.2 If the underlying process is from the stationary ARMA(p,q) model (1), then, for $j = 0, 1, \cdots,$

$$\hat{\phi}^{(j)}_{k,m} = \hat{\phi}^{(j)}_{m(k)} + o_p(n^{-1/2}), \qquad m = 1, \cdots, k.$$

Proof It is proved by mathematical induction on j. The result for $j = 0$ is proved in Lemma 4.1. Suppose that the theorem is true for j=0, $\cdots$, h. For j=h+1, equations (4) and (7) imply that

$$\sqrt{n}\,(\hat{\phi}^{(h+1)}_{k,m} - \hat{\phi}^{(h+1)}_{m(k)})$$

$$= \sqrt{n}\,(\hat{\phi}^{(h)}_{k,m} - \hat{\phi}^{(k)}_{m(k)}) - \sqrt{n} \left\{ \frac{\hat{\phi}^{(h)}_{k,m-1} \hat{\phi}^{(h)}_{k+1,k+1}}{\hat{\phi}^{(h)}_{k,k}} - \frac{\hat{\phi}^{(h)}_{m-1(k)} \hat{\phi}^{(h)}_{k+1(k+1)}}{\hat{\phi}^{(h)}_{k(k)}} \right\}$$

$$= o_p(1) + \{(\alpha^{(h)}_{k} + o_p(1)) \sqrt{n}\, \hat{\phi}^{(h)}_{k,m-1} - (\alpha_{k(h)} + o_p(1)) \sqrt{n}\, \hat{\phi}^{(h)}_{m-1(k)}\}$$

$$= \alpha^{(h)}_{k} \sqrt{n}\, \{\hat{\phi}^{(h)}_{k,m-1} - \hat{\phi}^{(h)}_{m-1(k)}\} + o_p(1),$$

where $\alpha_k^{(h)}$ is the one defined in Corollary 4.1. Hence, the result also holds for $j = h + 1$.

Corollary 4.2 Under the condidtions of Theorem 4.2, if $k \geq p$ and $j \geq q$, then $\sqrt{n}\,\hat{\phi}_{k,m}^{(j)}$ and $\sqrt{n}\,\hat{\phi}_{m(k)}^{(j)}$ have the same asymptotic distribution.

V. The Derivation of the Algorithm (7) for the EYW Estimates

In this section, we will not use the superimposed superscript '^' for notational conveniences unless there exists any possibility of confusion. That is, we will use the notations of parameters instead of their estimates.

We define a k by k Toeplitz matrix and two vectors as

$$B(k, j) = \begin{bmatrix} \rho_j & \rho_j & \cdots & \rho_{j-k+1} \\ \rho_{j+1} & \rho_{j-1} & \cdots & \rho_{j-k+2} \\ \vdots & \vdots & & \vdots \\ \rho_{j+k-1} & \rho_{j+k-2} & \cdots & \rho_j \end{bmatrix},$$

$$\underline{\rho}(k, j) = (\rho_{j+1}, \cdots, \rho_{j+k})', \text{ and}$$

$$\underline{r}(k, j) = (\rho_{j-k}, \cdots, \rho_{j-1})'$$

We denote $(x_n, \cdots, x_2, x_1)'$ by $\tilde{\underline{x}}$ for any vector $\underline{x} = (x_1, x_2, \cdots, x_n)'$.

Since the process $\{Z_t\}$ is not purely deterministic, the estimate $\det\{\hat{B}(k, j)\}$ is nonzero with probability one for any fixed n. Thus, we can define, for $k = 1, 2, \cdots$, and $j = 0, 1, \cdots$,

$$\underline{\phi}(k, j) = B(k, j)^{-1}\,\underline{\rho}(k, j),$$

$$\underline{\pi}(k, j) = B(k, j)^{-1}\underline{r}(k, j), \text{ and}$$

$$\lambda(k, j) = \rho_j - \tilde{\underline{\rho}}(k, j)' B(k, j)^{-1}\underline{r}(k, j).$$

For $j = 0, 1, \cdots$, we let $\lambda(0, j) = \rho_j$. We denote the j-th elements of $\underline{\phi}(k, j)$ and $\underline{\pi}(k, j)$ by $\phi_{k,j}^{(j)}$ and $\pi_{k,j}^{(j)}$, respectively. Also, we let $\phi_{k,0}^{(j)} = \pi_{k,0}^{(j)} = -1$. If we denote $\underline{\phi}_{(k+1,j-1)}$ by $(\underline{x}', y)'$, then

$$\begin{bmatrix} \tilde{\underline{r}}(k, j)', & \rho_{j-k-1} \\ B(k, j), & \underline{r}(k, j) \end{bmatrix} \begin{bmatrix} \underline{x} \\ y \end{bmatrix} = \begin{bmatrix} \rho_j \\ \underline{\rho}(k, j) \end{bmatrix},$$

and $y = \phi^{(j-1)}_{k+1,k+1}$. It implies that

$$\begin{aligned} \underline{x} &= B(k,j)^{-1}\underline{\rho}(k,j) - B(k,j)^{-1}\underline{r}(k,j)y \\ &= \underline{\phi}(k,j) - \underline{\tilde{\pi}}(k,j)\phi^{(j-1)}_{k+1,k+1}. \end{aligned}$$

If we let $\pi^{(j)}_{k,m} = -\phi^{(j-1)}_{k,k-m}/\phi^{(j-1)}_{k,k}$, then the following two equations are equivalent :

$$\begin{aligned} &B(k,j-1)\underline{\phi}(k,j-1) = \underline{\rho}(k,j-1), \text{ and} \\ &B(k,j)\,\underline{\tilde{\pi}}(k,j) = \underline{r}(k,j). \end{aligned}$$

It completes the derivation of Equation (7).

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