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The Backward Extended Yule-Walker Equation and the Order Determination of ARMA Processes

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A stationary ARMA process satisfies the (forward) extended Yule-Walker equations, which have been utilized in choosing optimal orders of the process. In this paper we present an associated nonstationary ARMA model that has the same autocorrelation function as the stationary one, and present the backward extended Yule-Walker equations of the associated ARMA process. Using the backward equations we present a new function whose pattern is useful to determine the orders of the ARMA process. Also, a recursive algorithm is presented to solve the forward and backward extended Yule-Walker equations simultaneously.

I. Introduction

Recently the spectrum estimation using ARMA models has been popularly used [2], [4], [9]. One of the most difficult problems in ARMA spectrum estimation is choosing optimal orders of ARMA models, which is the topic of this paper.

Consider the autoregressive moving-average (ARMA) model of orders p and q,

$$\phi(B)y_t = \theta(B)v_t \qquad (1)$$

where $\phi(z) = -\phi_0 - \phi_1 z - \cdots - \phi_p z^p$, $\theta(z) = -\theta_0 - \theta_1 z - \cdots - \theta_q z^q$, $\phi_0 = \theta_0 = -1$, $\phi_p \neq 0$, $\theta_q \neq 0$, B is the backshift operator i.e., $B^k y_t = y_{t-k}$ and $\{v_t\}$ is a white noise process with variance σ^2. Assume that all the roots of the characteristic equation $\phi(z) = 0$ are greater than 1 in absolute value, and assume that $\phi(z)$ and $\theta(z)$ have no common zeros. Let $\{\sigma(j)\}$ and $\{\rho_j\}$ be the autocovariance function (ACVF) and the autocorrelation function (ACRF)

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of the process defined by

$$\sigma(j)=Cov(y_t, y_{t+j}) \text{ and } \rho_j=\sigma(j)/\sigma(0)$$

It is well-known that the ACRF of the stationary ARMA(p, q) process (1) satisfies the following forward extended Yule-Walker equations (FEYW);

$$\rho_j=\emptyset_1\rho_{j-1}+\cdots+\emptyset_p\rho_{j-p} \qquad (2)$$
$$j=q+1, q+1, \cdots$$

Many functions of two variables, whose patterns are useful to decide orders of mixed ARMA processes, have presented by several authors [1], [4]~[7], [10], [12]~[15]. The patterns are due to the FEYW, and the associated order-determination methods are called the pattern identifications. It is known that there are other ARMA models than (1), each of which has the same ACRF as (1). In this paper we formulate one of the models that has the backward extended Yule-Walker equations (BEYW). Using the BEYW we define a new function whose pattern is also useful for choosing optimal orders of mixed ARMA processes. Also a simplified Trench-Zohar algorithm is presented to solve the FEYW and the BEYW simultaneously and recursively. Some numerical examples are given to illustrate usefulness of the new pattern.

Ⅱ. A Corresponding Backward Process

Consider an associated ARMA model to (1),

$$\pi(F)y_t=\theta(F)v_t \qquad (3)$$

where $\pi(z)=z^p\emptyset(z^{-1})/\emptyset_p$ and F is the forward-shift operator, i.e., $F^k y_t=y_{t+k}$. Denote $\pi(z)=-\pi_0-\pi_1 z-\cdots-\pi_p z^p$. Then, $\pi_0=-1$ and $\pi_p\neq 0$. Clearly the ARMA model (3) is nonstationary.

If the spectrums of the processes $\{y_t\}$ and $\{z_t\}$ are denoted by $S_y(\lambda)$ and $S_z(\lambda)$ respectively, then it can be easily shown that

$$S_z(\lambda)=\emptyset_p^2 S_y(\lambda), \qquad -\pi\leq\lambda\leq\pi$$

Thus the Wiener-Khinchine relation implies that the two processes have the same

ACRF$\{\rho_j\}$.

Since a root of $\phi(z)=0$ is greater than 1 in absolute value and is the reciprocal of that of $\pi(z)=0$, the backward process (3) can be represented as follows.

$$\pi_p\phi(B)y_{t+p}=\theta(B^{-1})v_t$$

$$\Leftrightarrow \quad y_{t+p}=\{\pi_p\phi(B)\}^{-1}\theta(B^{-1})v_t=\sum_{m=0}^{\infty}c_m v_{t+q-m}$$

where $\{c_m\}$ is a sequence of coefficients and the infinite series in the RHS converges in L^2 or almost surely. Therefore,

$$y_t=\sum_{m=0}^{\infty}c_m v_{t+(q-p)-m} \tag{4}$$

Equations (3) and (4) imply the BEYW;

$$\rho_j=\pi_1\rho_{j+1}+\cdots+\pi_p\rho_{j+p} \tag{5}$$
$$j=q-p+1,\ q-p+2,\ \cdots$$

Ⅲ. The η Function and Its Pattern

When the true ACRF is given, we solve the p FEYW (2) of $j=q+1,\ \cdots,\ q+p$ to obtain the forward AR parameters $\phi_1,\ \cdots,\ \phi_p$. However, if we use other p FEWY of $j=q+1,\ \cdots,\ q+i+p-1$ for any $i\in\{2,\ 3,\ \cdots\}$, then we obtain the same solution. Glasbey [6] proposed a function, which we will call the θ function;

$$\theta(p,\ i)=\rho_{i+p+1}-\phi_1\rho_{1+p}-\cdots-\phi_p\rho_{i+1} \tag{6}$$

The FEYW implies that $\theta(p,\ i)=0$ if $i=q,\ q+1,\ \cdots$. Moreover, it is well-known [3] that $\theta(q,\ p)\neq 0$. The θ pattern has been used to decide orders of ARMA processes. The same idea may be applied to the BEYW, because, for $i\in\{1,\ 2,\ \cdots\}$, the p BEYW (5) of $j=q-p+i,\ \cdots,\ q-1+i$ have the same solution of the backward AR parameters $\pi_1,\ \cdots,\ \pi_p$. Now we define a function called the η by

$$\eta(p,\ i)=\rho_{i-p-1}-\pi_1\rho_{i-p}-\cdots-\pi_p\rho_{i-1} \tag{7}$$

The BEYW implies that $\eta(p,\ i)=0$ if $i>q+1$. The η pattern is also helpful to choose optimal orders of ARMA processes.

To extend the θ and the η functions to the domain $\{(k,\ i) : k,\ i=0,\ 1,\ \cdots\}$, we first define some matrices and vectors. For a positive integer k, we define a k by Toeplitz matrix $\boldsymbol{B}(k,\ i)$ by

$$\boldsymbol{B}(k,\ i) = \begin{bmatrix} \rho_i & \rho_{i-1} & \cdots & \rho_{i-k+1} \\ \rho_{i+1} & \rho_i & \cdots & \rho_{i-k+2} \\ \vdots & \vdots & \cdots & \vdots \\ \rho_{i+k-1} & \rho_{i+k-2} & \cdots & \rho_i \end{bmatrix}$$

Denote its determinant by $d(k,\ i)$, and define two vectors by

$$\boldsymbol{\rho}(k,\ i)=(\rho_{i+1},\ \cdots,\ \rho_{i+k})' \text{ and } \boldsymbol{r}(k,\ i)=(\rho_{i-k},\ \cdots,\ \rho_{i-1})'$$

Henceforth we denote $(x_n,\ \cdots,\ x_2,\ x_1)'$ by $\tilde{x}$ for any vector $\boldsymbol{x}=(x_1,\ x_2,\ \cdots,\ x_n)'$.

Now we present the formal definitions of the θ and η functions and their related ones.

Definition For each pair of $(k,\ i)$, ($k=1,\ 2,\ \cdots$, and $i=0,\ 1,\ \cdots$), if $\boldsymbol{B}(k,\ i)$ is nonsingular, then we define

$$\boldsymbol{\phi}(k,\ i)=\boldsymbol{B}(k,\ i)^{-1}\,\boldsymbol{\rho}(k,\ i)$$
$$\tilde{\boldsymbol{\pi}}(k,\ i)=\boldsymbol{B}(k,\ i)^{-1}\,\boldsymbol{r}(k,\ i)$$
$$\theta(k,\ i)=\rho_{i+k+1}-\tilde{\boldsymbol{\phi}}(k,\ i)'\,\boldsymbol{r}(k,\ i)$$
$$\eta(k,\ i)=\rho_{i-k-1}-\boldsymbol{\pi}(k,\ i)'\,\boldsymbol{r}(k,\ i)$$

and

$$\lambda(k,\ i)=\rho_i-\tilde{\boldsymbol{\phi}}(k,\ i)'\,\boldsymbol{r}(k,\ i)$$

For $i=0,\ 1,\ \cdots$, we let $\theta(0,\ i)=\rho_{i+1}$, $\eta(0,\ i)=\rho_{i-1}$ and $\lambda(0,\ i)=\rho_i$.

As discussed before, the FEYW imply that the θ function of an ARMA process has a special pattern. More precisely, the process $\{y_t\}$ has an ARMA $(p,\ q)$ representation if and only if the pair p and q are the smallest integers satisfying $\theta(k,\ i)=0$ for any ($k=p$ and $1\geq q$) and for ($k\geq p$ and $i=q$) [6]. Similarly the BEYW imply a useful pattern of the η function as follows.

Theorem If the process $\{y_t\}$ has an ARMA$(p,\ q)$ representation, then the η function satisfies the following.

(a) $\eta(p-1, i)\neq 0, \qquad i=q+1, q+2, \cdots$

(b) $\eta(p, q+1)\neq 0$

(c) $\eta(p, i)=0, \qquad i=q+2, q+3, \cdots$

Its proof is in Appendix.

Since the vectors $\boldsymbol{\phi}(k, i)$ and $\boldsymbol{\pi}(k, i)$ need the inverse of the Toeplitz matrix $\boldsymbol{B}(k, i)$, Trenchs' algorithm about Toeplitz matrix inversion will be useful [15]. The following algorithm is a simplified version of Trench–Zohar's one.

Algorithm If $\boldsymbol{B}(k, i)$ is nonsingular, then

$$\lambda(k+1, i)=\lambda(k, i)-\theta(k, i)\eta(k, i)/\lambda(k, i)$$

Moreover, if the nonsingularity of $\boldsymbol{B}(k+1, i)$ is also assumed, then

$$\boldsymbol{\phi}(k+1, i)=\begin{bmatrix} \boldsymbol{\phi}(k, i)-\theta(k, i)\tilde{\boldsymbol{\pi}}(k, i)/\lambda(k, i) \\ \theta(k, i)/\lambda(k, i) \end{bmatrix}$$

$$\boldsymbol{\pi}(k+1, i)=\begin{bmatrix} \pi(k, i)-\eta(k, i)\tilde{\boldsymbol{\phi}}(k, i)/\lambda(k, i) \\ \eta(k, i)/\lambda(k, i) \end{bmatrix}$$

Ⅳ. Experiments and Comments

When a T-realization $\{y_1, \cdots, y_T\}$ is observed, we estimate ρ_1 by

$$\widehat{\rho_1}=\sum_{t=1}^{T-1}(y_{t+1}-\bar{y})(y_t-\bar{y})\Big/\sum_{t=1}^{T}(y_t-\bar{y})^2$$

where $\bar{y}=\sum y_{t+1}/T$. The other parameters and functions are estimated by substituting the sample ACRF for the true ACRF.

The first example is from the ARMA (2, 1) model;

$$(1+0.3B-0.1B^2)y_t=(1-0.7B)v_t$$

We use the true ACRF of the process. The η pattern in 〈Table 1〉 shows definitely

〈Table 1〉 The η array of the process $(1+0.3B-0.1B^2)y_t=(1-0.7B)v_t$

[redacted]					
0	1.000	-.6831	.3050	-.1598	.0784
1	0.781	-.5304	-.1012	-.0206	-.0041
2	1.597	3.148	0.000	0.000	0.000
3	2.361	Infi	3.148	0.000	0.000
4	3.429	Infi	Infi	3.148	0.000
5	4.937	Infi	Infi	Infi	3.148
6	7.081	Infi	Infi	Infi	Infi

Note : The word 'Infi' stands for the infinity.

〈Table 2〉 Empirical mean and standard deviation of η array of $(1-.95B)$ $y_t=(1-.4B)v_t$ with 50 sample size and 100 replications

[redacted]						
0	1.000	.829	.780	.733	.689	.647
	(0.000)	(.043)	(.054)	(.064)	(.073)	(.081)
1	-.380	.118	-.000	-.002	-.000	-.000
	(.109)	(.035)	(.019)	(.020)	(.019)	(.018)
2	-.997	-2.81	.118	.004	-.027	.000
	(.325)	(87.7)	(.376)	(.488)	(.428)	(.790)
3	-2.92	-1.43	-3.26	.110	-.255	-.011
	(10.2)	(38.1)	(118.)	(.510)	(8.51)	(.382)
4	-.778	-1.61	-.851	4.81	.116	.038
	(57.8)	(22.5)	(11.0)	(11.0)	(4.65)	(1.55)

Note : Figures within the parentheses are standard deviations.

that the underlying process is from an ARMA (2, 1) model. The second sample is from the ARMA (1, 1) model;

$$(1-.95B)y_t = (1-.4B)v_t$$

Newbold and Bos [11] analyzed the model through simulation experiments to show the unstability of Woodward and Gray's generalized partial autocorrelation (GPAC), which was proposed to determine orders of ARMA process [14]. To illustrate the large-sample stability of the estimate of $\eta(p, i)$, $i>q+1$, we generate 1000 replications of the ARMA(1, 1) process with length 500. And then, we tabulate the empirical means and standard deviations of the elements of the estimated η array in 〈 Table 2 〉

Since the empirical means of $\hat{\eta}(1, i)$'s, $i=3, 4, \cdots$, are very small in absolute value (near 0.000), and since their empirical standard deviations are small (near 0.019), we may say that the η pattern is stable for large sample cases. Also, the smallness of the empirical biases and variances makes us expect the consistencies of the η function estimates. Actually we can show that the consistency of the sample ACRF implies that of $\hat{\eta}(p, i)$ for $i=q+1, q+2, \cdots$. The above examples show that the η function is a useful tool of determining orders of mixed ARMA processes at least for large sample cases.

⟨APPENDIX⟩

Proof of Theorem If $\boldsymbol{B}(k, i)$ is nonsingular, then the determinant of $\boldsymbol{B}(k+1, i-1)$ is

$$\begin{aligned} d(k+1, i-1) &= det\begin{bmatrix} \tilde{\boldsymbol{r}}(k, i) & \rho_{i-k-1} \\ \boldsymbol{B}(k, i) & \boldsymbol{r}(k, i) \end{bmatrix} \\ &= (-1)^k det\begin{bmatrix} \rho_{i-k-1} & \tilde{\boldsymbol{r}}(k, i)^t \\ \boldsymbol{r}(k, i) & \boldsymbol{B}(k, i) \end{bmatrix} \\ &= (-1)^k det[\boldsymbol{B}(k, i)]det[\rho_{i-k-1} - \tilde{\boldsymbol{r}}(k, i)^t \boldsymbol{B}(k, i)^{-1}\boldsymbol{r}(k, i)] \\ &= (-1)^k det[\boldsymbol{B}(k, i)]det[\rho_{i-k-1} - \tilde{\boldsymbol{r}}(k, i)^t \tilde{\boldsymbol{\pi}}(k, i)] \\ &= (-1)^k d(k, i)\, \eta(k, i) \end{aligned}$$

If the underlying process is from an ARMA (p, q) model, then the rank of $\boldsymbol{B}(k, i)$ is $k-\min(k-p, i-q)$ for $k \geq p$ and $i \geq q$ [12]. This property completes the proof. Q.E.D

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