한국경제학보 제26권 제2호
The Korean Journal of Economics Vol. 26, No. 2 (Autumn 2019)

Discussion on Pareto Efficiency and Self-Stability of Weighted Majority Rules

Semin Kim* · Uzjune Kim**

Abstract

We consider a Bayesian environment with independent private values and two possible alternatives. We characterize the set of weak Pareto efficient weighted majority rules. Based on the characterization, we discuss the connection between Pareto efficiency and self-stability of weighted majority rules.

KRF Classification : C020700, B030200
Keywords : Pareto Efficiency; Self-Stability; Weighted Majority Rule

Ⅰ. Introduction

Pareto efficiency is a classic concept of efficiency. In a set of voting rules, one rule is Pareto efficient, if there exists no other rule that makesall agents better off with at least one strictly better off. This efficiency is weak that many economists just consider it as a necessary axiom. In this paper, we will examine this axiom through the lens of stability. Suppose members of a society can propose to replace the current rule by others, which rules will survive in the long-run? Can Pareto efficient rule remain? In the literature of the

* First Author, Associate Professor, School of Economics, Yonsei University, e-mail: seminkim@yonsei.ac.kr
** Second and Corresponding Author, Master's Student, School of Economics, Yonsei University, e-mail: uzpulkim@ gmail.com

stability of voting rules, Barberàand Jackson (2006) (BJ henceforth) develop a theory that is based on the endogenous preferences of agents over voting rules, a theoretical framework to study these questions. The key concept, which characterizes voting rules that can survive proposed changes, is called *self-stability*. As an example, a rule f is self-stable if, given any proposed alternative rule g, the coalition of agents who prefer g to f is not large enough to win the vote on replacing f by g, where the rule used for this latter decision is f itself. The idea underlying this concept is that the voting rule f governs both ordinary decisions and decisions on changes to the voting rule itself. BJ also consider the case of constitutions, where a different rule F is used to decide between f and g. This concept of constitution is used in this paper. In BJ's paper, they analyze self-stability of voting rules that treat all the voters symmetrically. In other words, they study the case of anonymous rules in which a reform passes if and only if the number of its supporters exceeds the threshold specified by the rule. They find that self-stable rules may not exist in this setup, and they establish conditions (on the characteristics of the society) that guarantee the existence self-stable rules.

Azrieli and Kim (2016) extend this analysis to a larger class of voting rules, namely to the class of weighted majority rules. There are many examples of institutions that use rules in which different agents have different voting weights: In the United Nations Security Council the permanent members have veto power, which gives them stronger voting power than the power of other members; in the Council of the European Union the number of seats of each country depends on its size; and in the International Monetary Fund members' voting weights depend on the size of their economies. Therefore, understanding what types of rules are self-stable is important, when anonymity is not assumed (One can find in Azrieli and Kim (2014) that those rules are indeed weighted majority rules).

Azrieli and Kim (2016)show that only few rules of a very particular form are self-stable. Each self-stable rule partitions the society into at most three groups, where the weights of agents within each group are identical. The first group contains 'veto' players,agents that can single-handedly vote to turn down any reform. The second group contains 'null' players whose votes are weighted zero, never affecting the outcome. The last group contains 'normal' players that represent agents of the rest of the society. According to a self-stable rule, a reform passes if and only if the coalition of reform supporting agents contains all the veto players and at least a certain number of normal players. There are additional constraints on the numbers of veto players and normal players that is necessary to be satisfied for the rule to be self-stable. These constraints vary with the characteristics of the society, but in every society self-stability implies that the rule has the form of at most three groups, described above.

On the other hand, Azrieli and Kim (2014) characterize the set of ex-ante and interim incentive efficient voting rules. The ex-ante stage is the information stage where no voter knows the value for alternatives, given the distribution of the values. On the other hand, the interim stage is the information stage where each voter's value for the alternatives is private. In this paper, the ex-ante stage is the main focus. The ex-ante incentive efficient voting rule is the rule that is (ex-ante) Pareto efficient and incentive compatible. It turns out to be a weighted majority rule with the special weights and quota. Regarding the Pareto efficiency of weighted majority rules, we use their arguments and results.

On top of the contributions of those papers above, this paper discusses the relationship between Pareto efficiency and Self-stability of weighted majority rules. In our formal model, a voting rule is any mapping from preference profiles over $\{Reform, Statusquo\}$=$\{R, S\}$ to lotteries over this set. A weighted majority rule is a voting rule that assigns weights to the agents and sets a quota, such that R is chosen if the total weight of the agents that support R exceeds the quota, and S is chosen if the total weight is smaller than the quota.[1)]

Our Proposition 1 is a characterization of the class of weakly Pareto efficient weighted majority rules. Here a weakly Pareto efficient rule is the rule that there is no other weighted majority rule that makes all agents strictly better off than before. In other words, this rule cannot be defeated by any other weighted majority rule in the sense that all agents do not strictly prefer any alternative weighted majority rule to this rule. Thus, it is naturally linked to the definition of self-stable constitution where the rule F is the unanimous rule. Proposition 1 provides this link between weak Pareto efficiency and self-stability of weighted majority rules.

Our Proposition 2 shows that the set of weak Pareto efficient weighted majority rules are indeed the same as the set of Pareto efficient weighted majority rules. The set are not the same in general; however, if we consider the set of ordinal weighted majority rules in our environment, Proposition 2 shows that two sets turn out to be the same. Combining two propositions, we are able to obtain the connection between Pareto efficiency and self-stability of weighted majority rules.

The assumptions we make about the environment (two alternatives, independent private values) are restrictive; however, they allow us to get a clean characterization of the Pareto frontier. From a practical point of view, the two alternatives assumption is less problematic, since binary decision is frequently used in reality.

1) It is slightly different from the weighted majority rules in Azrieli and Kim (2014) in the sense that their tie-breaking rule favors the status quo. This paper allows any tie-breaking rule.

Although type independence is restrictive in our context, given the pervasiveness of this assumption in the literature, we think that it is an interesting benchmark to study.

Ⅱ. Related Liturature

The theoretical study of weighted majority rules dates back at least to von-Neumann and Morgenstern's book (1944). The old literatures try to examine measurements of the power of players in various cooperative games generated by different rules (e.g., Shapley and Shubik, 1954).

The large body of this paper is also heavily related on mechanism design. The main difference between this paper and ours is the assumption. We assume that monetary transfer is not allowed. We are confident that such assumption is quite realistic because in everyday life for ethical, or even for moral reasons monetary transfers are infeasible or excluded.

Now, our paper is related to previous literatures like following. The model we use, in which agents' preferences over voting rules are endogenously determined from their assessments regarding their future preferences over alternatives, was first suggested in early papers by Rae (1969), Badger (1972), and Curtis (1972). In these papers, all agents' votes are weighted identically under anonymous voting rules.

The theoretical investigation of weighted majority rules appears already in the seminal book of von-Neumann and Morgenstern (1944), who are mainly interested in measures of the voting power of agents under the rule. A common scenario leading to heterogeneous voting weights is that of a representative democracy with heterogeneous district sizes. Azrieli and Kim (2014) show that, in a standard mechanism design setup, weighted majority rules naturally arise from considerations of efficiency and incentive compatibility.

Several papers extend the analysis of BJ's self-stability concept in various directions. Wakayama (2002) studies self-stability under the possibility that agents can abstain from voting. Kultti and Miettinen (2009)study self-stability in a model of constitutions with several layers of voting rules, where the voting rule in each layer is used to decide on changes to the voting rule of the previous one. The same authors consider a different set up such that there exists a continuum of agents. In this set up, they analyze stability of voting rules (Kultti and Miettinen, 2007)

Finally, the idea that the same voting rule used for the ordinary decision is also used for the special decision to choose between voting rules resembles the concept of self-selection for social choice functions introduced by Koray (2000). Also refer Barberà and Bevià (2002) and Koray and Slinko (2008).

The rest of paper is organized as follows. In the next section we describe the voting environment and give the definition of weighted majority rules. We mostly follow the notations in Azrieli and Kim (2016). In Section 4, we provide the characterization of weak Pareto efficient weighted majority rules. The characterization helps to discuss the connection between Pareto efficiency and self-stability of weighted majority rules. Proofs are in the Appendix.

Ⅲ. Enviroment

A society faces a binary decision of whether to implement a Reform (R) or to keep the Status quo (S). There are $n \geq 2$ agents in the society indexed by $i \in [n] := \{1, \ldots, n\}$. Each agent can either prefer R or S. The ex-ante probability that agent i prefers R is

$p_i \in (0,1)$, and with the complement probability $1-p_i$ he prefers S. We assume throughout the paper that agents' types are independent, and for every subset of agents (coalition) $T \subseteq [n]$ we denote

$$p(T) = \prod_{i \in T} p_i \prod_{i \notin T} (1-p_i)$$

the probability that the agents in T are exactly those who like R. Each agent i has the utility function, parameterized by $r_i > 0$: If R is implemented, then an agent who prefers R gets a utility of r_i and an agent who prefers S gets a utility of -1; if the Status quo prevails then both types get a utility of zero. A society can therefore be characterized by the pair $(\underline{p}, \underline{r})$, where $\underline{p} = (p_1, \ldots, p_n)$ and $\underline{r} = (r_1, \ldots, r_n)$.

A *voting rule* is used to aggregate the preferences of the agents into a decision. Formally, a voting rule is any mapping $f: 2^{[n]} \rightarrow [0,1]$, with the interpretation that, for any coalition T, $f(T)$ is the probability that R is chosen when the members of T are those who prefer R. Given a voting rule f, the expected utility of agent i is given by

$$U_i(f) = r_i \sum_{\{T: i \in T\}} p(T) f(T) - \sum_{\{T: i \notin T\}} p(T) f(T). \qquad (1)$$

In this paper we focus our attention on weighted majority rules. These are relatively simple voting rules and we refer the reader to Azrieli and Kim (2014) for a discussion of the importance of these rules based on efficiency considerations. The formal definition is as follows.

Definition 1. A voting rule f is a *weighted majority rule* if there are non-negative weights $\underline{w}=(w_1,...,w_n)$ and a quota $0<q<\Sigma_i w_i$ such that

$$f(T)=\begin{cases}1, \ if \sum_{i\in T} w_i > q \\ 0, \ if \sum_{i\in T} w_i < q\end{cases}$$

Let W be the set of weighted majority rules. Since the weighted majority rules in this paper are ordinal, those rules are incentive compatible, which is shown in Azrieli and Kim (2014). Thus, we can leave apart from the incentive problems and focus on the aspect of efficiency and stability. We write $f=(\underline{w},q)$ if f can be represented by these weights and quota.2) Given a weighted majority rule $(\underline{w},q)$, $w(T):=\Sigma_{i\in T} w_i$ denotes the total weight of coalition T. Finally, if $f=(\underline{w},q)$ then the expected utility of i under f in (1) can be rewritten as

$$U_i(f)=r_i \sum_{\{T: w(T)>q, i\in T\}} p(T) - \sum_{\{T: w(T)>q, i\notin T\}} p(T). \tag{2}$$

Ⅳ. Pareto Efficiency and Self-Stability

In thissection, we discuss the connection between Pareto efficiency and Self-stability. First, we define Pareto efficiency of voting rule.

Definition 2. A voting rule f is *Pareto Efficient* if there is no $g\in W$ such that $U_i(g)\geq U_i(f)$ for every agent i with at least one strictly

2) Note the weights and quota that define a weighted majority rule are typically not unique, that is the same rule f may be represented by different sets of weights and quotas. The concepts we study do not depend on the particular representation used.

better off.

For the direct connection between Pareto efficiency and Self-stability, we need a weak version of Pareto efficiency of voting rules.

Definition 3. A voting rule f is *weakly Pareto Efficient* if there is no $g \in W$ such that $U_i(g) > U_i(f)$ for every agent i.

Next, we define self-stability of weighted majority rules. Let f be a weighted majority rule and let g be an arbitrary voting rule. Denote by $T(g,f) = \{i \in [n] : U_i(g) > U_i(f)\}$ the coalition of agents for which rule g yields a strictly higher expected utility than rule f. We can now define the concept of self-stability.

Definition 4. Given a society $(\underline{p}, \underline{r})$, a weighted majority rule f is *self-stable* if $f(T(g,f)) = 0$ for any weighted majority rule g. Equivalently, if $f = (\underline{w}, q)$, then self-stability means that $w(\{i : U_i(g) > U_i(f)\}) < q$ for any weighted majority rule g.

In words, self-stability of an incumbent rule f means that no alternative rule g (the reform) would have sufficient support to replace f if the voting rule used to determine the winner is f itself. We emphasize that our definition of self-stability requires the incumbent rule f to be a weighted majority rule and allows the alternative rule g only to be weighted majority rules.[3)]

It is often the case that the voting rule used for everyday decisions, say f, is different than the rule used to make procedural amendments such as replacing f by another rule. Following BJ, we

3) It is also different from Azrieli and Kim (2016). This paper does not need the larger set of alternative rules than the weighted majority rules.

call a pair of weighted majority rules (f,F) a constitution. The interpretation is that F is the rule used to determine whether f will be replaced by another rule, and f is used for all other decisions. Then we can define self-stability of constitution in the following way.

Definition 5. Given a society $(\underline{p},\underline{r})$, a constitution $(f,F)\in W\times W$ is self-stable if $F(T(g,f))=0$ for every voting rule $g\in W$.

Note that this definition generalizes the definition of self-stable voting rule. We can define the self-stable voting rule, using the concept of constitution such that f is self-stable if and only if the constitution (f,f) is self-stable.

Now we are ready to show our main results.

Proposition 1. The following statements are equivalent.

1. *A voting rule* $f\in F$ *is weakly Pareto efficient.*
2. *A constitution* (f,F) *is self-stable where* F *is the unanimous rule.*
3. *There are non-negative numbers* $\lambda_1,\ldots,\lambda_n$ *such that* f *is a weighted majority rule with weights and quota given by* $(\lambda_1(r_1+1),\ldots,\lambda_n(r_n+1);\sum_{i\in N}\lambda_i)$.

This proposition first shows the relations between the weak Pareto efficiency and self-stability of weighted majority rules. Indeed, a weak Pareto efficient rule is a rule of constitution f under the unanimous rule of F. This connection is straightforward by the definitions of weak Pareto efficiency and self-stability. But it can show the stability of weak Pareto efficient rules in the sense that they endure the challenge of other weighted majority rules, not just show some degree of efficiency. Furthermore, this proposition characterizes the weak Pareto efficient rules. In Appendix, we also characterize the Pareto efficient rules where the only difference is that weak Pareto efficiency allows zero for λ_i for some agent i, not all. Thus, it looks that the set of weak Pareto efficient weighted majority rules are larger than the set of Pareto efficient weighted majority rules. However, the following proposition says that the sets are indeed the same.

Proposition 2. *A voting rule $f \in W$ is weakly Pareto efficient if and only if it is Pareto efficient.*

This is interesting by its own because Pareto efficiency implies weak Pareto efficiency in general, not the opposite direction. However, in our environment Proposition 2 shows that the opposite direction holds. Also, the following corollary can connect the concept of Pareto efficiency and self-stability of weighted majority rules combining Proposition 1 and 2. Thus the proof is omitted.

Corollary 1. A voting rule $f \in W$ is Pareto efficient if and only if a constitution (f, F) is self-stable where F is the unanimous rule.

Received: October 17, 2018. Revised: April 18, 2019. Accepted: December 10, 2019.

◈ *References* ◈

Azrieli, Y. and S. Kim (2014), "Pareto Efficiency and Weighted Majority Rules," *International Economic Review*, 55(4), 1067-1088.

Azrieli, Y. and S. Kim (2016), "On the Self-(in)Stability of Weighted Majority Rules," *Games and Economic Behavior*, 100, 376-389.

Badger, W. W. (1972), *Political Individualism, Positional Preferences, and Optimal Decision Rules*, In Probability Models of Collective Decision Making, edited by R.G. Niemi and H.F. Weisberg.

Merrill, Columbus, Ohio.

Barberà, S. and C. Bevià (2002), "Self-Selection Consistent Functions," *Journal of Economic Theory*, 105, 263-277

Curtis, R. B. (1972), *Decision Rules and Collective Values in Constitutional Choice*, In Probability Models of Collective Decision Making, edited by R.G. Niemi and H.F. Weisberg. Merrill, Columbus, Ohio.

Holmström, B. and R. B. Myearson (1983), "Efficient and Durable Decision Rules with Incomplete Information," *Econometrica*, 51, 1799-1819.

Jackson, M. O. (1991), "Bayesian Implementation," *Econometrica*, 59, 461-477.

Jackson, M. O. and H. F. Sonnenschein (2007), "Overcoming Incentive Constraints by Linking Decisions," *Econometrica*, 75, 241-257.

Kultti, K. and P. Miettinen (2007), "Stable Set and Voting Rules," *Mathematical Social Sciences*, 53, 164-171.

Kultti, K. and P. Miettinen (2009), "Stability of Constitutions," *Journal of Public Economic Theory*, 11, 891-896.

Koray, S. (2000), "Self-Selective Social Choice Functions Verify Arrow and Gibbard-Sattherthwaite Theorems," *Econometrica*, 68, 981-995.

Koray, S. and A. Slinko (2008), "Self-Selective Social Choice Functions," *Social Choice and Welfare*, 31, 129-149.

Rae, D.W. (1969), "Decision-rules and Individual Values in Constitutional Choice," *The American Political Science Review*, 63, 40-56.

Shapley, L. S. and M. Shubik (1954), "A Method for Evaluating the Distribution of Power in a Committee System," *American Political Science Review*, 48, 787-792.

von-Neumann, J. and O. Morgenstern (1944), *Theory of Games and Economic Behavior*, Princeton University Press.

Wakayama, T. (2002), "Endogenous Choice of Voting Rules with Abstention," Working paper.

Appendix

Proof of Proposition 1

The equivalence between 1and 2 are made by the definition of weakly Pareto efficiency and Self-stability of voting rules. We focus on the equivalence between 1 and 3.[4] First, we characterize the set of voting rules that maximize (ex-ante) social welfare, i.e., the sum of ex-ante expected utilities of all the agents in our environment.

Definition 6. The *(ex-ante) social welfare* of a voting rule f is

$$V(f) = \sum_{i \in N} U_i(f)$$

Lemma 1. A voting rule f is a maximizer of V if and only if it is the weighted majority rule with $\tilde{w} = r_i + 1$ for all i and $\tilde{q} = n$.

Proof of Lemma 1.

We refer readers to Theorem 1 in Azrieli and Kim (2014). In this lemma, we do not consider incentive compatibility, which does not affect the lemma. The weights and quota are derived according to our environment.

Assume first that f is a weighted majority rule with weights and quota as in the proposition. We will consider an auxiliary environment with the same set of agents, types and distribution over types as in the original environment. But the utilities in the new environment are given by $u_i'(t_i) = \lambda_i u_i(t_i)$ for every i and t_i. From Lemma 1, f is a maximizer of V in the auxiliary environment. In other words, f is a maximizer of

4) In this part, the proof is similar to the proof of Theorem 3 in Azrieli and Kim (2014). However, this proposition deals with weak Pareto efficiency, not (strong) Pareto efficiency in Azrieli and Kim (2014).

$$\sum_i U_t'(g) = \sum_i \lambda_i U_i(g)$$

among all functions $g \in W$. Since all the $\lambda s'$ are non-negative, f maximizes a linear combination with non-negative coefficients of the ex-ante utilities of the players. This proves that f is weakly Pareto efficient.

Conversely, let f be weakly Pareto efficient. We argue first that there are *non-negative* numbers $\{\lambda\}_{i \in N}$ such that f is a maximizer of

$$\sum_i \lambda_i U_i(g)$$

among all functions $g \in W$. Indeed, the set W is convex since it allows the randomized rules. Also, the mapping from voting rules to ex-ante utility vectors is affine: $U_i(\alpha f + (1-\alpha)g) = \alpha U_i(f) + (1-\alpha) U_i(g)$ for any $f, g \in W$ and any $\alpha \in [0,1]$. It follows that ex-ante utility possibility set is convex. For convex sets, weak Pareto efficiency is characterized by maximization of linear combinations of utilities with non-negative coefficients. Thus, f is socially optimal in an auxiliary environment with utilities given by $u_i'(t_i) = \lambda_i u_i(t_i)$. By Lemma 1, f satisfies the condition in the proposition. □

Proof of Proposition 2

[If part] It is obvious by definitions of weak Pareto efficiency and Pareto efficiency.

[Only if part] The following lemma can be derived from Theorem 3 in Azrieli and Kim (2014) where the weights and quota are changed corresponding to our environment. Thus, we omit the proof of Lemma 2.

Lemma 2. A voting rule $f \in W$ is Pareto efficient if and only if there are strictly positive numbers $\lambda_1, \ldots, \lambda_n$ such that f is a weighted majority rule with weights and quota given by

$$\lambda_1(r_1+1), \ldots, \lambda_n(r_n+1); \sum_{i \in N} \lambda_i$$

Assume that f is weakly Pareto efficient. By Proposition 1, we have non-negative numbers $\lambda_1, \ldots, \lambda_n$ such that f is a weighted majority rule with weights and quota given by $(\lambda_1(r_1+1), \ldots, \lambda_n(r_n+1); \Sigma_{i \in N} \lambda_i$. Let O be the set of agents whose $\lambda_i = 0$. Recall the form of weighted majority rules,

$$f(T) = \begin{cases} 1, \ if \sum_{i \in T} w_i > q \\ 0, \ if \sum_{i \in T} w_i < q \end{cases}$$

By Lemma 2, it is sufficient to show that there are strictly positive numbers $\lambda_1^{'}, \ldots, \lambda_n^{'}$ such that f is a weighted majority rule with new weights and quota given by
$(\lambda_1^{'}(r_1+1), \ldots, \lambda_n^{'}(r_n+1); \Sigma_{i \in N} \lambda_i^{'})$. Consider $\lambda_i^{'} = \lambda_i$ for $i \notin O$ and $\lambda_i^{'} = \varepsilon$ for $i \in O$. Then

$q' = q + \Sigma_{i \in O} \varepsilon$ and $\Sigma_{i \in T} w_i{}' = \Sigma_{i \in T-O} \lambda_i(r_i+1) + \Sigma_{i \in T \cap O} \varepsilon = \Sigma_{i \in T} w_i + \Sigma_{i \in T \cap O} \varepsilon$.

When $\Sigma_{i \in T} w_i > q, \Sigma_{i \in T} w_i{}' = \Sigma_{i \in T} w_i + \Sigma_{i \in T \cap O} \varepsilon > q \Sigma_{i \in T \cap O} \varepsilon = q' - \Sigma_{i \in O-T} \varepsilon$.

Then we can find extremely small $\varepsilon > 0$ that $\Sigma_{i \in T} w_i{}' > q'$ when $\Sigma_{i \in T} w_i > q$. The same logic can be applied when $\Sigma_{i \in T} w_i < q$. □

가중 다수결 제도의 파레토 효율성과 자체 지속성에 관한 논의

김 세 민* · 김 우 중**

논문초록

본 논문은 독립적이며 개인적인 가치를 가지는 두개의 대안 사이에서 투표권자들이 투표를 하는 상황을 고려합니다. 투표권자들은 자신의 가치는 알고 있으며 다른 투표권자들의 가치는 알지 못하나 그 확률적 분포는 알려져 있는 정보 환경을 상정하고 있습니다. 이론적 논의를 통하여 본 논문은 약 파레토 효율성을 가진 가중 다수 결제도란 무엇인지를 밝혔으며 나아가 가중 투표 제도의 파레토 효율성과 자체 지속성의 연관성을 밝혔습니다.

주제분류 : C020700, B030200
핵심 주제 : 파레토 효율성, 자체 지속성, 가중 다수결 제도

* 제1저자, 연세대학교 경제학부 부교수, e-mail: seminkim@ yonsei.ac.kr
** 제2저자 및 교신저자, 연세대학교 경제학부 석사, e-mail: uzpulkim@gmail.com