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Optimal Forecasting Using Bivariate Recursive Structural VAR Model

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Abstracts

We suggest the optimal forecasting of bivariate vector autoregressive (VAR) process when one of predicted variable is available. A sufficient condition for this predictor to be more efficient than the forecasting just using reduced form is that the structural VAR (SVAR) model is recursive. As a test statistic to check the recursiveness of SVAR, we suggest the Hausman test. According to the simulation, suggested predictor improved the forecasting efficiency.

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Ⅰ. Introduction

After Sims (1980), VAR models have been widely used and successful in dynamic analyses including the forecasting and impulse response analysis, Granger causality test and forecasting error variance decomposition. For instance in forecasting, as Stock

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and Watson (p6, 1993) correctly asserted, one of the most important advances in forecasting methodology after the 1980s was the development and refinement of forecasting with small vector autoregressions.

The VAR model assumes the data are observed at the same time. However, in practice, it is not realistic and some of them are observed in advance than the other variables. The focal point of this paper in forecasting is the possible lag in the availability of the data. For instance, variables like the interest rates and stock prices, are available without informational lag. In contrast, the GDP has the information lag of almost one full quarter. See Estrella and Mishkin (1998).

In this paper, we suggest the optimal forecasting of bivariate VAR process when one of predicted variable is available. A sufficient condition for this predictor be more efficient than the forecasting using reduced form is that the SVAR model is recursive. As a test statistic to check the recursiveness of SVAR, we suggest the Hausman test. According to the simulation, suggested predictor improved the forecasting efficiency well.

The rest of the paper is organized as follows. In Section 2, we discuss on the forecasting with the recursive SVAR Model. In Section 3, the efficient estimation of optimal predictor and test of recursiveness is introduced. Section 4 is on the Monte Carlo simulation results and remained research.

Throughout the paper, standard notations are used without any explicit references. In particular, we use $\xrightarrow{p}$ and $\xrightarrow{d}$ to signify the convergence in probability and in distribution, respectively, of random sequences.

Ⅱ. Recursive Structural VAR Model

Let $z_t \equiv (x_t, y_t)'$ denote a 2×1 vector of mean-adjusted variables, assumed to be generated by the r-th order SVAR process,

$$\begin{aligned} A_0 z_t &= A_1 z_{t-1} + A_2 z_{t-2} + \cdots + A_r z_{t-r} + \varepsilon_t \\ &\equiv A Z_{t-1} + \varepsilon_t \end{aligned} \tag{1}$$

for $t = 0, 1, \cdots, n$ where

$$A_0 \equiv \begin{pmatrix} 1 & \alpha_{12} \\ \alpha_{21} & 1 \end{pmatrix}$$

with the normalization, $A_0, A_1, \cdots, A_r$ are 2×2 matrices of autoregressive coefficients and $\varepsilon_t \equiv (v_t, w_t)'$ is a 2×1 vector of unobserved mean zero independent, identically distributed (iid) disturbances with a positive definite matrix

$$\Sigma \equiv \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix},$$

$A \equiv (A_1, A_2, \cdots, A_r)$ and $Z_{t-1} \equiv (z'_{t-1}, z'_{t-2}, \cdots, z'_{t-r})'$.

The reduced form of (1) may be written as

$$\begin{aligned} z_t &= B_1 z_{t-1} + B_2 z_{t-2} + \cdots + B_r z_{t-r} + \epsilon_t \\ &\equiv B Z_{t-1} + \epsilon_t \end{aligned} \tag{2}$$

where the coefficient matrix A_0 is not singular, with $B_i \equiv A_0^{-1} A_i; i = 1, 2, \cdots, r,$ $B \equiv (B_1, B_2, \cdots, B_r)$ and $\epsilon_t \equiv A_0^{-1} \varepsilon_t$. Denote the first and second rows of coefficient B as $B_a (1 \times 2r)$

and $B_b(1\times 2r)$ respectively and thus $B\equiv (B_a{}', B_b{}')'$.

We assume the process (2) is stationary where all the roots of the characteristic equation of process,

$$|I_\ell\lambda^r - B_1\lambda^{r-1} - \cdots - B_r| = 0$$

have moduli less than one, that is, $|\lambda_i|< 1$ for $i=1,\cdots,2r$.

As in a usual SVAR model, we assume the coefficient A_0 has a triangular structure and the variance Σ is diagonal as:

Assumption 2.1 [*recursiveness*][1] (a) $\alpha_{12}=0$ and (b) $\sigma_{12}=0$.

For instance, the oil price may affect to the consumer price index (CPI) contemporaneously, however, the reverse causality may do not exist. This reflects the feature the raw material prices including oil are not determined by any specific country but by the international market.

Under Assumption 2.1, the first variable x_t is not affected by the contemporaneous shock to the variable y_t. More specifically, it may be readily shown that the second equation error of SVAR model, w_t, is not correlated with x_t under Assumption 2.1.

Proposition 2.2 *Suppose* Assumption 2.1 *holds. Then* w_t *is not correlated with* x_t.

Proof of Proposition 2.2 Note we may write

$$x_t = B_a Z_{t-1} + v_t$$

1) After Sims (1980), this assumption is standard to identify the SVAR model.

from definition. Therefore, note

$$cov(x_t, w_t) = B_a cov(Z_{t-1}, w_t) + cov(v_t, w_t) = 0$$

since Z_{t-1} is composed of lagged ε_t's and $\sigma_{12} = 0$ under Assumption 2.1. ■

For a moment, assume all VAR coefficients A are known for the simplicity.[2)] Conventional forecasting of z_t on Z_{t-1} is typically based on the reduced form VAR model (2). In particular, the conditional expectation based on Z_{t-1} is given as

$$E(z_t | Z_{t-1}) \equiv BZ_{t-1} \tag{3}$$

which is optimal in the sense of minimizing the forecasting mean squared error (FMSE). Especially, the FMSE of predictor (3) is given as;

$$A_0^{-1} \Sigma A_0^{-1\prime} = \frac{1}{(1-\alpha_{12}\alpha_{21})^2} \times \tag{4}$$

$$\begin{pmatrix} \alpha_{12}^2\sigma_{22} - 2\alpha_{12}\sigma_{12} + \sigma_{11} & -\alpha_{21}\sigma_{11} + (1+\alpha_{12}\alpha_{21})\sigma_{12} - \alpha_{12}\sigma_{22} \\ & \alpha_{21}^2\sigma_{11} - 2\alpha_{21}\sigma_{12} + \sigma_{22} \end{pmatrix}.$$

Sometimes one of the predicted variable, say, x_t is available in advance of the other variable y_t. For instance, financial variable including interest rate spread are available earlier than the gross domestic product (GDP).[3)] See Table in Appendix B on the

2) Feasible form of predictor with the estimated coefficient will be discussed in following section.

3) A series of papers in early 90's- Stock and Watson (1989b), Friedman and Kuttner (1992, 1993), Bernanke (1990), and Kashyap, Stein, and Wilcox

information lags of economic variables. In such a case, the information on the variable x_t may enhance the forecasting efficiency in terms of FMSE.

To show this, write the conditional expectation of y_t on x_t and Z_{t-1} using the SVAR model as

$$E(y_t | x_t, Z_{t-1}) \equiv -\alpha_{21} x_t + B_b Z_{t-1} \tag{5}$$

which results in the FMSE σ_{22}. Thus from (4), we show the predictor using the variable x_t additionally may reduce the FMSE as;

Theorem 2.3 *Suppose* Assumption 2.1 *holds. Then*[4)]

$$\text{FMSE}[E(y_t | Z_{t-1})] - \text{FMSE}[E(y_t | x_t, Z_{t-1})] = \alpha_{21}^2 \sigma_{11} \geq 0. \tag{6}$$

Note, most naturally, the FMSE difference in (6) depends on the size of α_{21} which signifies how the predicting variable x_t affects to the predicted variable y_t. The h-period ahead optimal forecasting of z_t is of course determined by the iterative application of reduced form VAR model (2) in usual fashion as;

(1993)-has shown that the spread between the respective interest rates on commercial paper and Treasury bills has borne a systematic relation to subsequent fluctuations of nonfinancial economic activity in the Unite States.

4) It is interesting that this kind of inequality does not hold in general without a recursiveness assumption. For a counter example, let $\alpha_{21} = 1$, $\sigma_{11} = \sigma_{22} = 1$ and $\sigma_{12} = 0$ where $-1 \leq \sigma_{12} \leq 1$ from Cauchy-Schwarz inequality. In this case, the difference of FMSE's in (6) becomes

$$\frac{2}{(1-\alpha_{12})^2} - 1 < 0$$

for the values of $\alpha_{12} < 1 - \sqrt{2}$ or $\alpha_{12} > 1 + \sqrt{2}$.

$$E(z_{t+1}|x_t, Z_{t-1}) \equiv B_1 E(z_t|x_t, Z_{t-1}) + B_2 z_{t-1} + \cdots + B_r z_{t-r+1}$$

$$E(z_{t+2}|x_t, Z_{t-1}) \equiv B_1 E(z_{t+1}|x_t, Z_{t-1}) + B_2 E(z_t|x_t, Z_{t-1}) + \cdots + B_r z_{t-r+2}$$

$$\vdots$$

$$E(z_{t+h}|x_t, Z_{t-1}) \equiv B_1 E(z_{t+h-1}|x_t, Z_{t-1}) + \cdots + B_2 E(z_t|x_t, Z_{t-1}) + \cdots + B_r z_{t-r+h}$$

where

$$E(z_t|x_t, Z_{t-1}) \equiv \begin{pmatrix} x_t \\ -\alpha_{21}x_t + B_b Z_{t-1} \end{pmatrix}.$$

Note the augmented efficiency due to the information on the variable x_t affects to the whole forecasted range till any period ahead.

Ⅲ. Efficient Estimation of Optimal Predictor and Test of Recursiveness

Now note the predictor (5) is not feasible since the coefficient is not known. However, under Assumption 2.1, we may estimate it consistently by the OLS regression. To show it, we rewrite the second equation of (1) as

$$\begin{aligned} y_t &= -\alpha_{21}x_t + B_b Z_{t-1} + w_t \\ &\equiv \delta' m_t + w_t \end{aligned} \tag{7}$$

where $\delta \equiv (-\alpha_{21}, B_b')'$ and $m_t \equiv (x_t, Z_{t-1}')'$. Then note the OLS estimator of the coefficient δ in the equation (7) is defined as;

$$\hat{\delta}_{ols} \equiv (m'm)^{-1}m'y \tag{8}$$

where $m \equiv (m_1, m_2, \cdots, m_n)'$ and $y \equiv (y_1, y_2, \cdots, y_n)'$. So the feasible predictor of y_t on x_t and Z_{t-1} becomes

$$\hat{E}(z_t | x_t, Z_{t-1}) \equiv \hat{\delta}'_{ols} m_t.$$

Now we discuss to test Assumption 2.1 using Hausman (1978) test. Denote the null and alternative hypotheses to be tested are given as;

$$H_0;\ \alpha_{12} = 0 \text{ and } \sigma_{12} = 0$$

$$\text{vs.}$$

$$H_A;\ \alpha_{12} \neq 0 \text{ of } \sigma_{12} \neq 0.$$

Under the null hypothesis, above OLS estimator $\hat{\delta}_{ols}$ is consistent and efficient from Gauss-Markov theorem, since the variable x_t is not correlated with the error w_t as shown in Proposition 2.2.

Under the alternative hypothesis, above OLS estimator has a limiting bias as;

$$\hat{\delta}_{ols} \xrightarrow{p} \delta + var(m_t)^{-1} cov(m_t, w_t)$$

where $cov(m_t, w_t) = -\alpha_{21}(\sigma_{12} - \alpha_{12}\sigma_{22})$.[5] To get a consistent estimator of the coefficient δ, an instrumental variable (IV) estimator may be constructed as;

5) We may approximate the covariance as $cov(m_t, w_t) \simeq \alpha_{21}\alpha_{12}\sigma_{22}$, if assuming $\sigma_{12} = 0$ seems to be satisfied more likely. According to Watson (1994, p2904 of *Handbook of Econometrics* Vol. IV), this covariance restriction might be fulfilled by introducing A_0 coefficient.

$$\hat{\delta}_{iv} \equiv (m'P_h m)^{-1} m'P_h z_2$$

where $q > r$, $h_t \equiv \left(z_{t-1}', z_{t-2}', \cdots, z_{t-r+1}', z_{t-q}'\right)'$, $h \equiv (h_1, h_2, \cdots, h_n)$, $P_h \equiv h(h'h)^{-1}h'$. Here the instruments h are composed of lagged VAR variables and thus are not correlated with the error w_t.

Both the OLS and IV estimators have following limit distributions;

Theorem 3.1 Suppose Assumption 2.1 holds. Then

$$n^{1/2}\left(\hat{\delta}_{ols} - \delta\right) \xrightarrow{d} N\left(0, avar\left(\hat{\delta}_{ols}\right)\right)$$

and

$$n^{1/2}\left(\hat{\delta}_{iv} - \delta\right) \xrightarrow{d} N\left(0, avar\left(\hat{\delta}_{iv}\right)\right)$$

where $var\left(\hat{\delta}_{iv}\right) \equiv \sigma_{22}\, plim\, n\left(w'P_h w\right)^{-1}$ *and* $var\left(\hat{\delta}_{ols}\right) \equiv \sigma_{22}\, plim\, n\left(w'w\right)^{-1}$.

Now the Hausman test statistic for the null hypothesis H_0 in usual fashion is defined as

$$\tau_n \equiv n\left(\hat{\delta}_{iv} - \hat{\delta}_{ols}\right)' \left[avar\left(\hat{\delta}_{iv}\right) - avar\left(\hat{\delta}_{ols}\right)\right]^{-1} \left(\hat{\delta}_{iv} - \hat{\delta}_{ols}\right)$$

where $\hat{\sigma}_{22}$ *is a consistent estimator of* σ_{22}, $avar\left(\hat{\delta}_{iv}\right) \equiv \hat{\sigma}_{22} n\left(w'P_h w\right)^{-1}$ *and* $avar\left(\hat{\delta}_{ols}\right) \equiv \hat{\sigma}_{22} n\left(w'w\right)^{-1}$.

Thus note (i) both $\hat{\delta}_{iv}$ and $\hat{\delta}_{ols}$ are consistent and asymptotically normally distributed, and (ii) the variance estimator $avar\left(\hat{\delta}_{ols}\right) - avar\left(\hat{\delta}_{iv}\right)$ is a consistent estimate of the variances' difference under the null hypothesis. So we may deduce following result

$$\tau_n \xrightarrow{d} \chi^2_{2r+1}$$

from Theorem 2.1 of Hausman (1978). Of course, the power of test comes from the difference of $\hat{\delta}_{iv} - \hat{\delta}_{ols}$ under the alternative.

Ⅳ. Monte Carlo Simulation and Remained Research

In this section, we conduct the Monte Carlo experiments to evaluate the performances of suggested predictor and Hausman test. Consider two dimensional VAR (2) model as a data generating process (DGP);

$$A_0 z_t = A_1 z_{t-1} + A_2 z_{t-2} + \varepsilon_t, \quad t = 1, 2, \cdots, 100 \tag{9}$$

where $z_t \equiv (x_t, y_t)'$ and the error was generated from $\epsilon_t \sim iidn(0, I_2)$,

$$A_0 \equiv \begin{pmatrix} 1 & \alpha_{12} \\ 0.3 & 1 \end{pmatrix}, \; A_0^{-1} A_1 \equiv \begin{pmatrix} 0.2 & -1.2\rho + 0.58 \\ 0.3 & 0.5 \end{pmatrix} \text{ and}$$
$$A_0^{-1} A_2 \equiv \begin{pmatrix} 0.2 & -0.1 \\ 0.2 & \rho + 0.1 \end{pmatrix}.$$

Under this frame work, we may check the process (z_t) has a unit root if $\rho = 1$ and it is stationary if $|\rho| < 1$ from the checking of characteristic roots. We conduct the 10,000 simulations for different values of ρ and α_{12}. Nonzero value of α_{12} implies the violation of recursiveness.

We suppose one of the predicted variable, say, x_t is available in advance of the other variable y_t. We first evaluate the FMSE of z_t at time t using the predictors with the reduced form VAR and SVAR respectively. Second, we compute the critical values of Hausman test statistic for the nominal size of 5%, 10%, 25% and 50% respectively.

According to simulation, our theoretical expectation has been confirmed. If $\alpha_{12} = 0$, then the SVAR approach was better than the predictor using reduced form VAR model in FMSE. See Table A in Appendix.

On the other hand, the Hausman test showed power if $\alpha_{12} = .3$ while the size was distorted.[6] See Table A in Appendix. So, to overcome this size distortion and lack of power stability in Hausman test, we may apply the graph-theoretic methods for causal analysis of Swanson and Granger (1997) or Demiralp and Hoover (2003). Demiralp and Hoover show how to apply graph-theoretic methods to selecting the causal order for a SVAR. To evaluate such approach in our forecasting suggestion is another promising research area.

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Appendix

[Table A] Simulation Results of FMSE and Hausman Test

		FMSE		empirical critical values[1]			
ρ	α_{12}	SVAR	VAR	50%	25%	10%	5%
0	0	1.060	1.157	0.903	1.568	2.390	2.970
	.3	0.971	1.385	0.944	1.608	2.422	3.030
0.1	0	1.081	1.166	0.980	1.692	2.505	3.038
	.3	0.977	1.373	1.001	1.713	2.570	3.186
0.2	0	1.079	1.157	1.057	1.800	2.618	3.230
	.3	0.967	1.374	1.026	1.784	2.663	3.363
0.3	0	1.035	1.113	1.072	1.810	2.691	3.303
	.3	0.970	1.359	1.076	1.866	2.800	3.434
0.4	0	1.053	1.141	1.088	1.841	2.738	3.378
	.3	0.963	1.366	1.083	1.861	2.845	3.502
0.5	0	1.134	1.227	1.125	1.880	2.774	3.480
	.3	0.952	1.349	1.074	1.856	2.812	3.623
0.6	0	1.225	1.322	1.092	1.830	2.748	3.403
	.3	0.980	1.408	1.090	1.846	2.819	3.490

1) on Hausman test

【Table B】 Indicator Series and Their Information Lags

Description	Information Lag (Months)
Interest Rates and Spreads	
10-year-3-month Treasury spread	0
Commercial paper-Treasury spread (6 months)	0
3-month T bill	0
10-year T bond	0
Stock Prices	
Dow Jones industrials	0
NYSE composite	0
S&P 500	0
Monetary Aggregates	
Monetary base	1
M1	1
M2	1
M3	1
Individual Macro Indicators	
Growth in real GDP, previous quarter	3
Consumer price index	1
Purchasing managers' survey	0
Vendor performance	0
Consumer and orders for plant and equipment	1
Housing permits	1
Consumer expectations	0
trade weighted dollar	0
Change in manufactures' unfilled durable goods orders	1

Source: Estrella and Mishkin (1998)

2변수 순차구조 VAR 모형을 이용한 최적예측

김 윤 영*

논문초록

본고는 2변수 VAR 모형에서 예측대상변수 중 하나가 미리 알려진 경우의 최적예측방법에 대해 논의하였다. 특히 구조 VAR 모형을 이용한 예측이 축약형 VAR 모형을 이용한 경우보다 효율적일 수 있는 충분조건은 구조 모형이 순차적이어야 함을 보였다. 여기서 구조 VAR 모형의 순차성을 검증하기 위한 통계량으로 본고는 Hausman 검정을 제시하고 있다. 모의실험에 따르면 제시된 예측방법은 예측 효율성을 제고시킴을 보였다.

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