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A Study on Sequental Testing Procedure for Outliers and Structural Change

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Abstracts

In this paper a test for outliers based on externally studentized residuals is shown to be related to a test for predictive failure. The relationships between a test for outliers, a test for a correlated mean shift and a test for an intercept shift are developed. A sequential testing procedure for outliers and structural change is shown to be independent, so that the overall size of the joint test can be determined exactly. It is established that a joint test for outliers and constancy of variances cannot be performed.

Keywords : Predictive Failure, Sequences of Independent Tests, A Test for Outliers, A Test for Structural Change

I. Introduction

The usefulness and reliability of linear regression models depend upon a set of assumptions regarding the choice of explanatory variables, the nature of the error term, the stability of the coefficients over time, and the ability to predict future observations accurately. Tests for coefficient stability and predictive failure have been used widely in econometrics since Chow (1960) presented the analysis of

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covariance and prediction error tests as alternative methods of checking for structural change. These tests look for certain departures from constancy of the regression coefficients when it is presumed that a structural break occures at a known point. Since structural change can arise through differences in the error variances as well as in the regression coefficients, the variance ratio test may also be used for testing the null hypothesis of a constant structure.

When the mean of the dependant variable is shifted for one or more observations but is otherwise left unspecified, a related diagnostic approach is to test for outliers. By a suitable re-ordering of the data, tests of predictive failure may be shown to be related to tests for outliers in the data. Such a re-interpretation enables a sequential testing procedure to be established for outliers and for structural change, and also permits a clarification of the issues invloved in performing the various tests.

The purpose of the paper is three-fold. First, a praticularly straightforward testfor single or joint outliers is shown to be related to a test for predictive failure. Second, the relationships between a test for outliers, a test for a correlated mean shift and a test for an intercept shift are developed. Third, a sequential testing procedure for outliers and structural change is shown to be independent, so that the size of the joint test can be determined exactly. A useful by-product of the analysis is the finding that it is not possible to test jointly for outliers and constancy of variances although it is well known that it is possible to test jointly for constancy of variances and regression coefficients.

The plan of the paper is as follows. In Section Ⅱ, the models are introduced and the tests for predictive failure, outliers and structural change are presented. The independence of the sequence of tests is established in Section Ⅲ. Some concluding remarks are given in Section Ⅳ.

Ⅱ. Notation and Test Statistics

The problem concerns the linear regression model

$$y_i = X_i\beta_i + u_i \qquad (i = 1,2)$$

in which y_i is $n_i\times 1$, X_i is an $n_i\times k$ matrix of observations on k non-stochastic regression, β_i is $k\times 1$ and u_i is $n_i\times 1$. The total number of observations available is $n=n_1+n_2$, where $n_1>k$ but n_2 is unrestricted. It is assumed that $u_i \sim N(0, \sigma^2 I_{n_i})$, in which I_{n_i} is the identity matrix of order n_i, so that the errors have constant variances throughout the analysis. The errors u_i and u_2 are also assumed to be uncorrelated. For purposes of considering an intercept shift, it is assumed there is an intersept in the model.

Consider the following formulation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ X_2 & I_{n_2} \end{bmatrix}\begin{bmatrix} \beta_1 \\ \delta \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1}$$

in which δ is $n_2\times 1$. If the observations are ordered naturally so that the first n_1 are used for estimation and the second n_2 are used for prediction, the above regression is equivalent to adding n_2 dummy variables taking the value of one for each of the observations n_1+1, $\cdots$, n_1+n_2. A test of $\delta=0$ is equivalent to testing $\beta_1=\beta_2$. When $n_2>k$, this has come to be known in the econometric literature as the Chow test. When n_2 is not restricted, a test of $\delta=0$ is a check for predictive failure and this is referred to as Chow's second test. If the second n_2 observations come from the same population as the first n_1, then the actual prediction errors should not differ significantly from zero. However, if the data have been re-ordered so that the set of observation given by n_2 are potential outliers, then a test of $\delta=0$ is a test for outliers using the mean-shift model (See, for example, Cook and Weisberg 1982, p. 20). Rejection of $\delta=0$ indicates there are n_2 outliers in the data. When $n_2=1$, a test of $\delta=0$ using prior information as to which observation is the potential outlier is equivalent to a test based on externally studentized residuals.

When $n_2>k$, consider a restricted version of (1) given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ X_2 & X_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \gamma \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2}$$

in which $\delta = X_2\gamma$ and $\gamma = \beta_2 - \beta_1$. Note that (2) is a reparameterization of

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

in which th regression models are estimated separately over the first n_1 and second n_2 observations. In (2), $\gamma=0$ is equivalent to $\beta_1=\beta_2$ and indicates there is no structural change. However, the restriction $\delta = X_2\gamma$ suggests a correlated mean shift which is dependent upon the columns of X_2. If interest centres on testing $\gamma=0$, this clearly implies $\delta=0$ and, since $n_2>k$, the reverse also holds. When neither δ nor γ equal zero, a test of the n_2-k linear restrictions $\delta = X_2\gamma$ is both straightforward and potentially informative. Rejection of $\delta = X_2\gamma$ indicates there are n_2 outliers in the data.

It is straightforward to show that the (unrestricted) residual sum of squares from (1) is equal to the residual sum of squares from regression y_1 on the columns of X_1 alone (See, for example, Salkever 1976). The test of $\delta=X_2\gamma$ is based on the statistic

$$\frac{(RSS-RSS_1)/(n_2-k)}{RSS_1/(n_1-k)}$$

in which the (restricted) residual sum of squares from (2) is given by $RSS = RSS_1 + RSS_2$, where $RSS_i = (y_i - X_i\hat{\beta}_i)'(y_i - X_i\hat{\beta}_i)$ is obtained from using n_i observation and $\hat{\beta}_i = (X_i'X_i)^{-1}X_i'y_i$ $(i = 1, 2)$. The test statistic may be rewritten as

$$\frac{RSS_2/(n_2 - k)}{RSS_1/(n_1 - k)} \sim F(n_2 - k,\ n_1 - k)$$

which will be recognized as the variance ratio (VR) test of equality of the error variances against the alternative that the variance of the n_2 (outlying) observations exceeds the variance of the n_1 observations. It should be noted, however, that equality of variances is maintained throughout the analysis, but rejection of $\delta = X_2\gamma$ may arise through non-constancy of variances as well as predictive failure or the existence of outliers. Since a test of $\delta = X_2\gamma$ is algebraically equivalent to the VR test, it is possible to test for either outliers or constancy of variances, but not both. Therefore, a joint test for outliers and constancy of variances cannot be performed.

A further restriction of equation (2) is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ X_2 & \ell \end{bmatrix}\begin{bmatrix} \beta_1 \\ \alpha \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{3}$$

in which ℓ is an $n_2 \times 1$ vector of unit elements and α is a scalar. Equation (3) denoted as intercept shift and is obtained from (2) by imposing the $k-1$ linear restrictions $X_2\gamma = \alpha\ell$. When $n_2 \leqq k$, the restriction $\delta = X_2\gamma$ cannot be tested but the $n_2 - 1$ linear restrictions $\delta = \alpha\ell$ can be tested, so that (3) may be obtained directly from (1). Rejection of $\delta = \alpha\ell$ indicates there are n_2 outliers in the data.

Finally, a test of the single restriction $\alpha = 0$ may be performed on (3) to yield the fully restricted model

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \beta + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4}$$

in which $\beta = \beta_1 = \beta_2$, and there is no structural shift in the model or outliers in the data. Note that a test of $\gamma = 0$ in (2) leads directly to (4), and this is the analysis of covariance (AOC) test for structural change. Since a test of $\delta = X_2\gamma$ or, equivalently, the VR test, is performed prior to the AOC test, it is well known that it is possible to test jointly for constancy of variances and regression coefficients. A schematic outline of the sequences of tests is given in the accompanying diagram.

Ⅲ. Sequences of Independent Test

It was shown above that the successively more restrictive models (1), (2), (3) and (4) were obtained by imposing linear restrictions on the previously maintained model. This leads to a set of seqquential tests for which the overall size must be determined. In the following propositions, it is shown that the overall size of the test can be controlled exactly because the sequences of tests are independent.

Proposition 1

When $n_2 > k$ and equation (4) is correct, the sequences of tests of $\delta = X_2\gamma$ and $\gamma = 0$, of $\delta = \alpha\ell$ and $\alpha = 0$, and of $\delta = X_2\gamma$, $X_2\gamma = \alpha\ell$ and $\alpha = 0$, are mutually independent.

⟨Proof⟩

The independence of the general test sequence of $\delta = X_2\gamma$, $X_2\gamma = \alpha\ell$ and $\alpha = 0$ will be shown. This sequence involves moving progressively from equations (1) through to (4). Denote the residual sum of squares from the $j'th$ equation as $u'M_ju$, in which $u = (u_1', u_2')'$ and M_j is the usual idempotent matrix based on the set of regression for the $j'th$ equation ($j = 1, 2, 3, 4$). Let $zj = u'(M_j - M_{j-1})u$, with $M_0 = 0$, where Z_j is the increase in the residual sum of squares when the restrictions are imposed on the $(j-1)'th$ equation to obtain the $j'th$ equation, for $j = 2, 3, 4$. Since $M_iM_j = M_i$ for $i \leqq j$, $M_j - M_{j-1}$ is idempotent and $(M_j - M_{j-1})(M_k - M - k - 1) = 0$ for $j \neq k$. It follows that the standardized variables Z_j/σ^2 are mutually independent and distributed as chi-squared. Since the sequence of F tests from equations (1) through to (4) are given by

$$\frac{Z_2/(n_2-k)}{Z_1/(n_1-k)}, \frac{Z_3/(k-1)}{(Z_1+Z_2)/(n-2k)} \quad \text{and}$$

$$\frac{Z_4}{(Z_1+Z_2+Z_3)/(n-k-1)},$$

it follows from the results of Hogg and Tanis (1963) that the three test statistics are mutually independent. The same method of proof may be applied to the test sequences of $\delta = X_2\gamma$, $\gamma = 0$ and $\delta = \alpha\ell$ and $\alpha = 0$.

Since the test of $\delta = X_2\gamma$ is algebraically equivalent to the VR test, and the test of $\gamma = 0$ is the AOC test, Proposition 1 provides an alternative proof of the independence of the VR and AOC tests (See Phillips and McCabe, 1983).

Proposition 2

Regardless of the value of n_2, when equation (4) is correct the test sequence of $\delta = \alpha\ell$ and $\alpha = 0$ is independent.

〈Proof〉

See the proof of Proposition 1.

Owing to the independence of the sequences of tests, if α_g denotes the size of the $g'th$ test, the overall size of the sequence of G tests is given by $1-\prod_{g=1}^{G}(1-\alpha_g)$. Thus, if the size of each test is held constant and $G=3$, then the size of each of the three tests should be approximately 1.7% if the overall size is required to be 5%.

Ⅳ. Concluding Remarks

It is common practice to test for predictive failure and structural change in econometrics. These tests are based on testing $\delta=0$ in going from equation (1) to equation (4) and, when $n_2>k$, on testing $\gamma=0$ to go from (2) to (4), respectively. In this paper we have established the independence of sequences of tests from equations (1), (2) and (3) through to (4) when $n_2>k$, and from (1) and (3) through to (4) when $n_2\leqq k$. Owing to this independence property, the overall size of the joint test may be obtained exactly and the possible rejection of the restrictions may be isolated more accurately. For example, rejection of $\delta=0$ in moving directly from equation (2) and (3), and not necessarily only with (1). Therefore, the sequential testing procedure advocated in this paper has potential informational value for applied researchers.

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