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A Meaningful Spin-off from Frisch/Waugh Theorem

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Based on Frisch/Waugh Theorem, we derive a simple analytical expression which explicates the exact determinant of correlation between a pair of OLS estimates of slope coefficients in the context of the standard multiple linear regression model.

I. Introduction

In theory of linear regression, a great deal of discussion is focused on the effect of the multicolinearity between regressors on the efficiency of the estimated coefficients but very little on the determinant of the correlation between a pair of OLS estimates of slope coefficients. This note is intended to fill the slight existing gap by deriving a concise analytical expression which brings out the core nature of correlations between estimates of slope coefficients using the partitioned representation of OLS estimator.

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II. Partitioned Representation of OLS Estimator

Consider the standard multiple linear regression model under the classical assumptions :

$$y = X\beta + u \tag{1}$$

where $X(n \times k) = [x_1, x_2, \cdots, x_k]$ is non-stochastic, of full column rank, and independent of the residual vector u an *iid* $(0, \sigma^2 I_n)$. We further assume that $x_1 = \lambda \equiv [1, 1, \cdots, 1]'$.

The linear model in (1) can be rewritten in arbitrarily but correspondingly partitioned X and β as :

$$Y = (X_A, X_B)\begin{pmatrix} \beta_A \\ \beta_B \end{pmatrix} + u \tag{2}$$

By virtue of what is known as Frisch/Waugh Theorem in Greene [2], the partitions in $b = (b_A{}'\ b_B{}')$, OLS estimator of β, corresponding to the partitions in $\beta = (\beta_A{}'\ \beta_B{}')'$ can be expressed as[1)]

$$b_A = (X_A{}' M_B X_A)^{-1} X_A{}' M_B\, y \tag{3}$$
$$b_B = (X_B{}' M_A X_B)^{-1} X_B{}' M_A\, y \tag{4}$$

1) The partitioned expression of OLS estimator is credited to a seminal paper by Frisch and Waugh [1] published in the inaugural volume of *Econometrica*, hence referred to as Frisch/Waugh Theorem. As a historical note, Ragnar Frisch (1895 ~ 1973) was one of the sixteen founding members of the Econometric Society (1930 ~), and served as the first Editor-in-Chief of *Econometrica* (1933 ~ 1955), and shared the first Nobel prize in Economics (1969) with Jan Tinbergen (1903 ~ 1994).

where

$$M_B = I_n - X_B(X_B{}'X_B)^{-1}X_B{}' \tag{5}$$

$$M_A = I_n - X_A(X_A{}'X_A)^{-1}X_A{}' \tag{6}$$

The respective covariance matrices for b_A and b_B easily follow from (3) and (4) respectively as :

$$\mathrm{Cov}(b_A) = \sigma^2(X_A{}'M_B X_A)^{-1} \tag{7}$$

$$\mathrm{Cov}(b_B) = \sigma^2(X_B{}'M_A X_B)^{-1} \tag{8}$$

Using the covariance matrix above for either b_A or b_B, we can derive the correlation coefficient between any pair of OLS estimates of slope coefficients as shown in Section Ⅲ.

Ⅲ. Derivation

Let X_A in (2) be defined as

$$X_A = (x_i\ x_j) \tag{9}$$

where $i, j = 2, 3, \cdots, k$ and $i \neq j$.

Then, M_B in (5) will be defined correspondingly as

$$M_B = \tilde{M}_{ij} \equiv I_n - \tilde{X}_{ij}(\tilde{X}_{ij}{}'\tilde{X}_{ij})^{-1}\tilde{X}_{ij}{}' \tag{10}$$

where $\tilde{X}_{ij}$ denotes an $n \times (k-2)$ matrix which consists of columns of X with x_i and x_j excluded.

Substituting (9) and (10) into (7),

$$\begin{aligned} \text{Cov}(b_A) &= \begin{bmatrix} \text{Var}(b_i) & \text{Cov}(b_i, b_j) \\ \text{Cov}(b_j, b_i) & \text{Var}(b_j) \end{bmatrix} \\ &= \sigma^2 [(x_i\, x_j)' \tilde{M}_{ij} (x_i\, x_j)]^{-1} \\ &= \sigma^2 \begin{bmatrix} x_i' \tilde{M}_{ij} x_i & x_i' \tilde{M}_{ij} x_j \\ x_j' \tilde{M}_{ij} x_i & x_j' \tilde{M}_{ij} x_j \end{bmatrix}^{-1} \\ &= \frac{\sigma^2}{\delta} \begin{bmatrix} x_j' \tilde{M}_{ij} x_j & -x_j' \tilde{M}_{ij} x_i \\ -x_i' \tilde{M}_{ij} x_j & x_i' \tilde{M}_{ij} x_i \end{bmatrix} \end{aligned} \tag{11}$$

where

$$\delta = (x_i' \tilde{M}_{ij} x_i)(x_j' \tilde{M}_{ij} x_j) - (x_i' \tilde{M}_{ij} x_j)^2$$

Reflecting the respective analytical solutions for $\text{Var}(b_i)$, $\text{Var}(b_j)$ and $\text{Cov}(b_i, b_j)$ derived in (11) on the definition of $\text{Corr}(b_i, b_j)$, the correlation coefficient between b_i and b_j,

$$\begin{aligned} \text{Corr}(b_i, b_j) &= \frac{\text{Cov}(b_i, b_j)}{\sqrt{\text{Var}(b_i)}\sqrt{\text{Var}(b_j)}} \\ &= -\frac{x_i' \tilde{M}_{ij} x_j}{\sqrt{x_i' \tilde{M}_{ij} x_i}\sqrt{x_j' \tilde{M}_{ij} x_j}} \end{aligned} \tag{12}$$

Noting that $\tilde{M}_{ij}$ in (9) is idempotent and symmetric and further that $x_i^* = \tilde{M}_{ij} x_i$ and $x_j^* = \tilde{M}_{ij} x_j$ are residual vectors resulting from regressions of x_i and x_j, respectively, on $\tilde{X}_{ij}$ (which contains $x_1 = \lambda$, therefore $\lambda' x_i^* = \lambda' x_j^* = 0$), the following quantities are by definition the covariance between and respective variances of x_i^* and x_j^* (x_i and x_j with the effect of $\tilde{X}_{ij}$ netted out or partialed out):

$$\frac{1}{n} x_i' \tilde{M}_{ij} x_j = \frac{1}{n} (\tilde{M}_{ij} x_i)'(\tilde{M}_{ij} x_j) = \frac{1}{n} x_i^{*\prime} x_j^* = \mathrm{Cov}(x_i^*, x_j^*)$$
$$\frac{1}{n} x_i' \tilde{M}_{ij} x_i = \frac{1}{n} (\tilde{M}_{ij} x_i)'(\tilde{M}_{ij} x_i) = \frac{1}{n} x_i^{*\prime} x_i^* = \mathrm{Var}(x_i^*)$$
$$\frac{1}{n} x_j' \tilde{M}_{ij} x_j = \frac{1}{n} (\tilde{M}_{ij} x_j)'(\tilde{M}_{ij} x_j) = \frac{1}{n} x_j^{*\prime} x_j^* = \mathrm{Var}(x_j^*) \qquad (13)$$

Reflecting (13) on (12),

$$\begin{aligned} \mathrm{Corr}(b_i, b_j) &= -\frac{\mathrm{Cov}(x_i^*, x_j^*)}{\sqrt{\mathrm{Var}(x_i^*)}\sqrt{\mathrm{Var}(x_j^*)}} \\ &= -\mathrm{Corr}(x_i^*, x_j^*) \end{aligned} \qquad (14)$$

The final analytical result above may well be summarized as: the correlation coefficient between OLS estimates of any pair of slope coefficients in the standard multiple linear regression model is identically equal in magnitude to, but in the opposite in sign of, the partial correlation coefficient between the corresponding regressors x_i and x_j.

Ⅳ. Conclusion

This note has derived a concise analytical expression which brings out the core nature of correlations among OLS estimates of slope coefficients in the context of the standard multiple linear regression model, facilitated by the Frisch/Waugh Theorem. In addition to its theoretical virtue, it may well of a pedagogical value.

▣ *References* ▣

1. Frisch, R. and F. Waugh, "Partial Time Regressions as Compared with Individual Trends," *Econometrica* 1, 1933, pp. 387～401.
2. Greene, W. H., *Econometric Analysis*, Fifth Edition, Prentice Hall, New Jersey, 2003, p. 27.